The Running Fix on the Sphere

… and Ellipsoid too!

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Outline

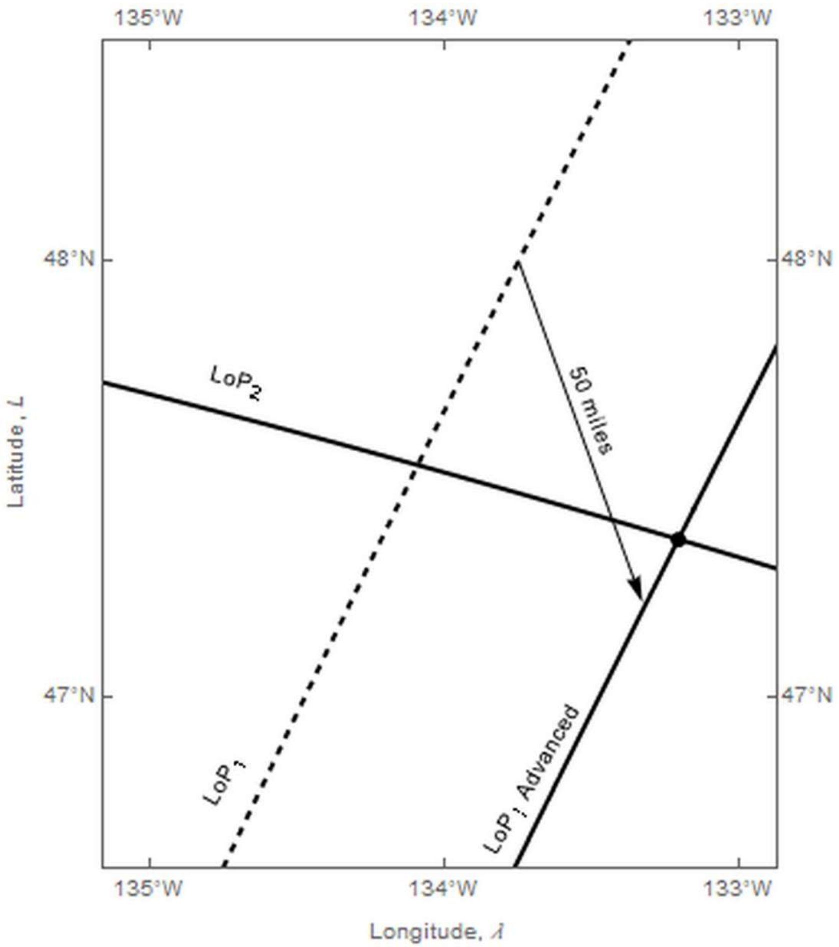
- Classic Running Fix on the Plane by advancing Line of Position

- Advancing Circles of Position on the Sphere and where things go wrong

- Solution to the problem of the Running Fix on the Sphere

- Generalization to the Running Fix on an Ellipsoid

Classic Running Fix



At 17:00:00 UT on 29 February, 2016 on a vessel located near 48°N a Sun sight measures ZD_1=77°36.8′

The Nautical Almanac gives GHA_1=71°54.3′; d_1=7°36.8′S

Line of Position (LoP_1) is plotted

Ship steams 50 nautical miles on a course 160°T
At 22:00:00 UT a Sun sight measures ZD_2=51°13.6′

The Nautical Almanac gives GHA_2=56°13.6′; d_2=7°32.1′S

Line of Position (LoP_2) is plotted

LoP_1 is advanced by the ship's course and distance travelled

Running Fix for position at 22:00:00 UT is point where Advanced LoP_1 crosses LoP_2

Circles of Position

Lines of Position are Circles of Position globally

<u>How should you advance a Circle of Position for a Running Fix?</u>

Advancing Circles of Position

Merrifield, J., 1886*, A Treatise on Nautical Astronomy*, Sampson Low, Marston, Searle and Rivington, London (Ex. 457)

- Adjusts *ZD* to account for run of ship

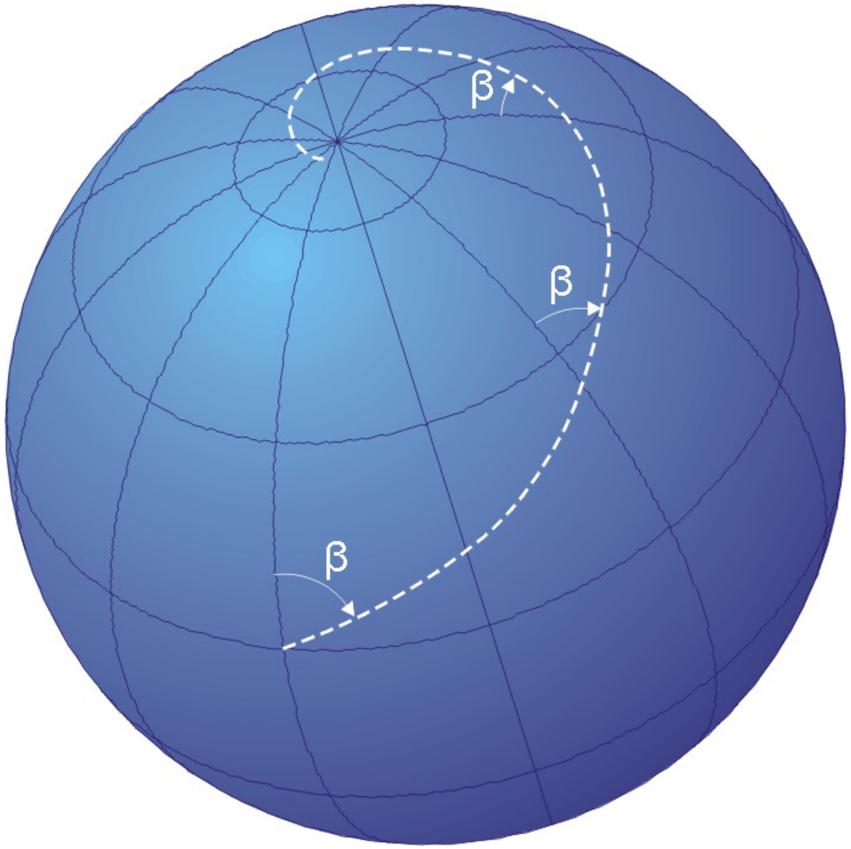
Metcalf, T. R. (1991) Advancing Celestial Circles of Position, *NAVIGATION: Journal of the Institute of Navigation*. **38**, 285–288.

> "…maintains the relationship between the geographical position of the body and the vessel, and hence is the best method for advancing an observation"

Zevering, K. H. (2006) Dependability of Position Solutions in Celestial Sight-Run-Sight Cases – Part 1 *The Journal of Navigation* (2006), **59**, 155–166

> "…only the GHA-Dec updating technique (GD-UT) will give a correct position solution"

Rhumbline on the Sphere



Points on a rhumbline follow a spiral path to the pole

Initial CoP is distorted as each point is advanced on the same course same course and distance

Ref: http://en.wikipedia.org/wiki/Rhumb_line

Advancing Circles of Position

Advanced CoP is <u>not</u> a circle

Any proposal for obtaining a Running Fix on the sphere assuming Advanced CoP's are circles
– is fundamentally flawed!

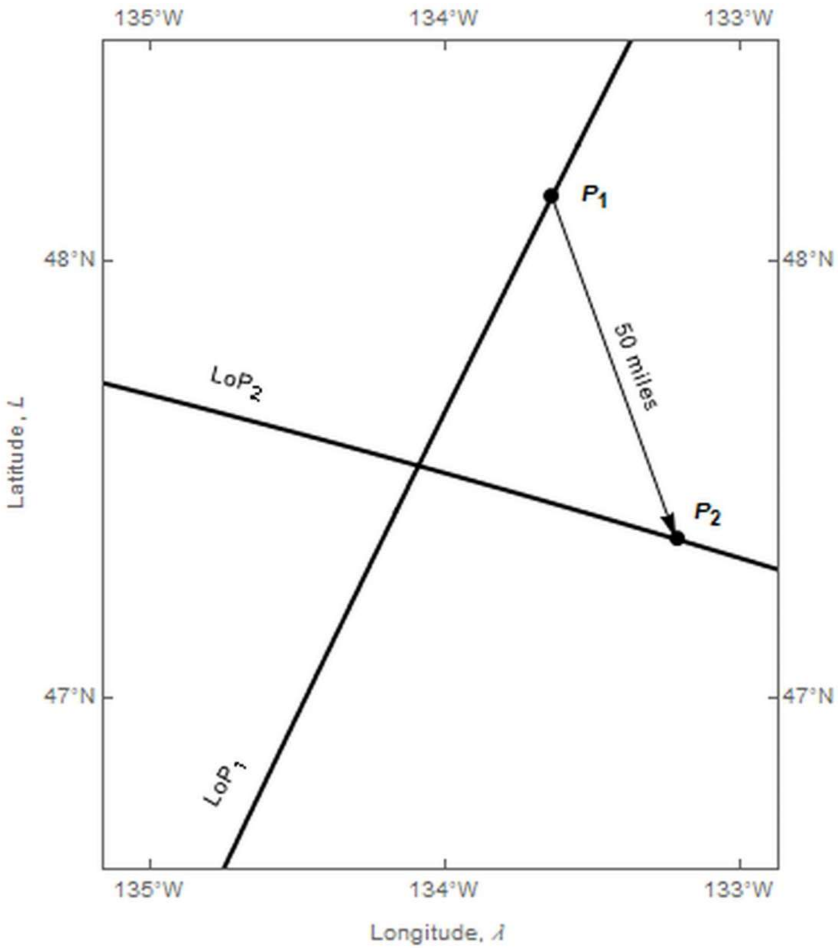
George Huxtable:
" On a sphere, if all points on a circle are displaced through the same distance and direction, the result is not a circle at all, but will have been distorted into a sort-of egg-shape."

Ref: Huxtable, G. (2006). An Erroneous Proposal To Allow For Travel Of An Observer Between Two Celestial Altitude Observations, *The Journal of Navigation*, **59**, 521–529.

Williams:
- Gives a (correct) exact expression for advanced CoP on an ellipsoid
- Suggests solving 2 simultaneous non-linear equations to obtain Running Fix.

Ref: Williams, R. (1998). *Geometry of Navigation*, Horwood Publishing Ltd, Chichester, UK.

The Running Fix on the Sphere



Label position at first sight be P_1 and at second sight, P_2

What do we know about be P_1 and P_2?

1. P_1 lies on LoP_1
2. P_2 lies a distance D from P_1 along a rhumb line course with bearing C.
3. P_2 lies on LoP_2

Conditions are sufficient to solve for be P_1 and P_2

- No need to advance LoP_1
- No need to solve simultaneous non-linear equations

Running Fix on the Sphere in Practice

Point P_1 with latitude L_1, longitude λ_1, satisfies the usual formula for altitude or Zenith Distance, ZD

$$\cos ZD_1 = \sin\delta_1 \sin L_1 + \cos\delta_1 \cos L_1 \cos\left(\lambda_1 + \mathrm{GHA}_1\right)$$

For a given latitude L_1, the corresponding longitude λ_2, can be found from

$$\lambda_1 = -\mathrm{GHA}_1 \pm \cos^{-1}\left(\frac{\cos ZD_1 - \sin\delta_1 \sin L_1}{\cos\delta_1 \cos L_1}\right)$$

Point P_2 with latitude L_2, longitude λ_2, lies distance D on course C from point P_1

$$L_2 = L_1 + D\cos C \qquad \text{or} \quad \text{D.Lat.} = D \cos C$$

$$\lambda_2 = \lambda_1 + \left(\mathrm{MP}\left(L_2\right) - \mathrm{MP}\left(L_1\right)\right)\tan C \qquad \text{or} \quad \text{D.Lon.} = \text{DMP} \tan C$$

Point P_2 with latitude L_2, longitude λ_2, satisfies the usual formula for altitude or Zenith Distance, ZD

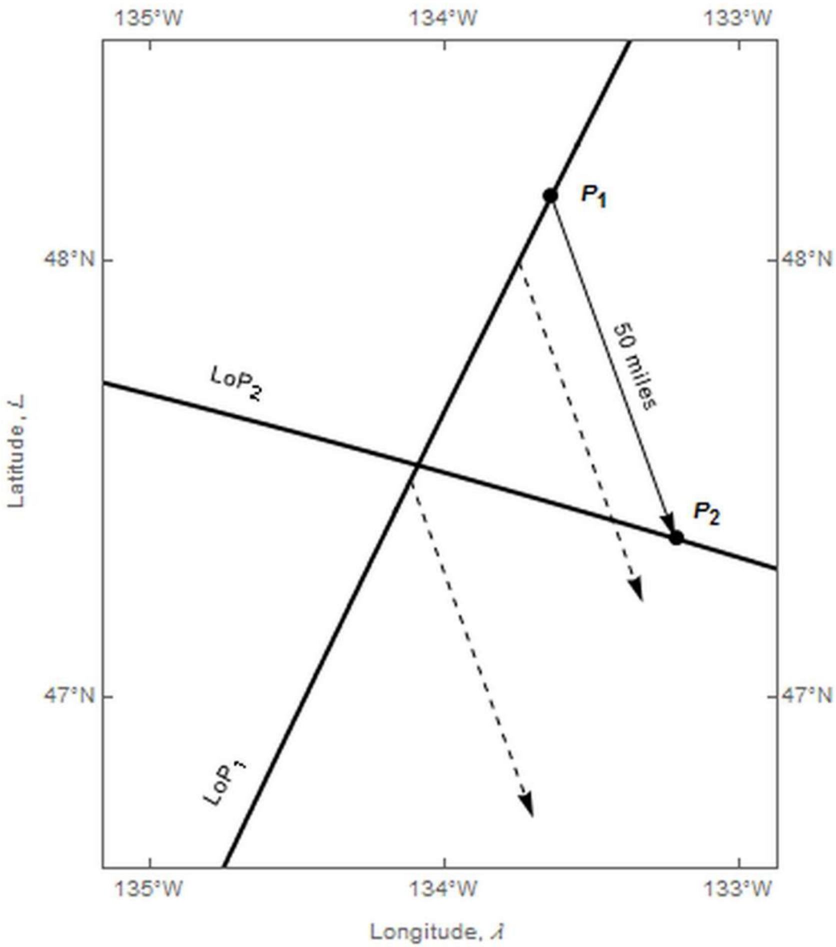
$$\cos ZD_2 = \sin\delta_2 \sin L_2 + \cos\delta_2 \cos L_2 \cos\left(\lambda_2 + \mathrm{GHA}_2\right)$$

We seek the value of L_1 for which

$$\mathrm{f}\left(L_1\right) \equiv \sin\delta_2 \sin L_2 + \cos\delta_2 \cos L_2 \cos\left(\lambda_2 + \mathrm{GHA}_2\right) - \cos ZD_2 = 0$$

i.e. Need to find the root of a function of 1 variable – an easy numerical problem

The Running Fix on the Sphere



Apply standard numerical methods to solve for $f(L_1) = 0$

Secant Method

- Bootstrap with $L_1 = 48°$ and $L_1 = 47.5°$
- 3 iterations produce 0.2 metre accuracy

Both P_1 and P_2 are obtained

Running Fix on the Ellipsoid in Practice

Point P_1 with latitude L_1, longitude λ_1, satisfies the usual formula for altitude or Zenith Distance, ZD

$$\cos ZD_1 = \sin\delta_1 \sin L_1 + \cos\delta_1 \cos L_1 \cos\left(\lambda_1 + \mathrm{GHA}_1\right)$$

For a given latitude L_1, the corresponding longitude λ_2, can be found from

$$\lambda_1 = -\mathrm{GHA}_1 \pm \cos^{-1}\left(\frac{\cos ZD_1 - \sin\delta_1 \sin L_1}{\cos\delta_1 \cos L_1}\right)$$

Point P_2 with latitude L_2, longitude λ_2, lies distance D on course C from point P_1

$$\mathrm{LP}\left(L_2\right) = \mathrm{LP}\left(L_1\right) + \left(\frac{D}{a}\right)\cos C \qquad \text{or} \qquad \mathrm{DLP} = D\cos C$$

$$\lambda_2 = \lambda_1 + \left(\mathrm{MP}\left(L_2\right) - \mathrm{MP}\left(L_1\right)\right)\tan C \qquad \text{or} \qquad \text{D.Lon.} = \mathrm{DMP}\tan C$$

Point P_2 with latitude L_2, longitude λ_2, satisfies the usual formula for altitude or Zenith Distance, ZD

$$\cos ZD_2 = \sin\delta_2 \sin L_2 + \cos\delta_2 \cos L_2 \cos\left(\lambda_2 + \mathrm{GHA}_2\right)$$

We seek the value of L_1 for which

$$\mathrm{f}\left(L_1\right) \equiv \sin\delta_2 \sin L_2 + \cos\delta_2 \cos L_2 \cos\left(\lambda_2 + \mathrm{GHA}_2\right) - \cos ZD_2 = 0$$

i.e. Need to find the root of a function of 1 variable – an easy numerical problem

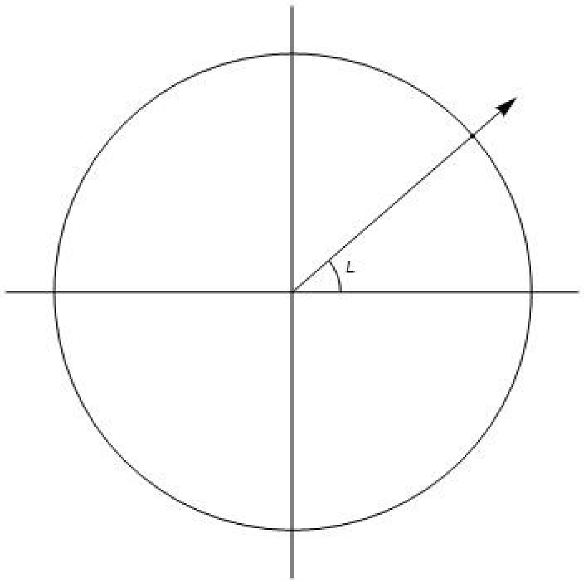
LoP on an Ellipsoid

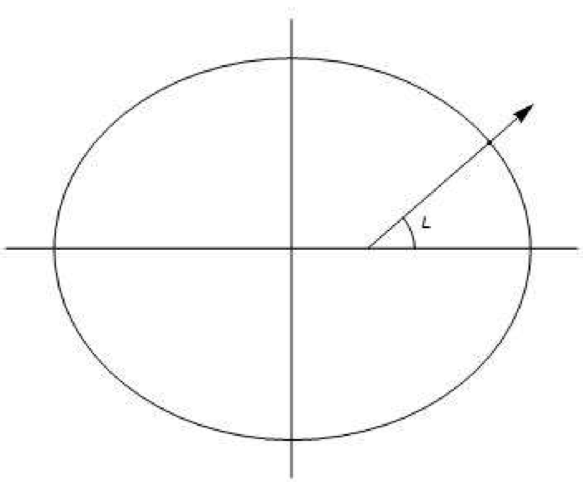
Equation for LoP is the same for the sphere and ellipsoid!

- Analytic proof given by Williams, see http://fer3.com/arc/m2.aspx/Constant-altitude-curves-an-ellipsoid-Stuart-aug-2015-g32468

Result is easily explained:

- LoP is a curve of constant Zenith Distance
- Direction of the vertical is the same at position latitude, L, longitude, λ, on a sphere or ellipsoid.





Latitudinal Parts

Latitudinal Parts function, LP(L), gives ellipsoidal arc length along a meridian from equator to latitude L

$$\mathrm{LP}(L) = \left(1-e^2\right)\int_0^L \left(1-e^2\sin^2\theta\right)^{-\frac{3}{2}} d\theta$$

- Units of the semi-major axis, a
- Can be expressed in terms of Legendre elliptic integrals
- For the sphere, $e = 0$ and LP(L) = L

Both LP and LP^{-1} needed

- Series expansions to $O(e^{10})$ in Tseng, W.-K., A. Earle, M. A. and Guo, J.-L. (2012). Direct and Inverse Solutions with Geodetic Latitude in Terms of Longitude for Rhumb Line Sailing. *The Journal of Navigation*, **65**, 549–559.
- When consistently expanded in the eccentricity, e

$$\mathrm{LP}(\phi) = \frac{\mathrm{E}(e)}{\pi/2}\phi - \left(\tfrac{3}{8}e^2 + \tfrac{3}{32}e^4 + \ldots\right)\sin 2\phi + \left(\tfrac{15}{256}e^4 + \ldots\right)\sin 4\phi + \ldots$$

$$\mathrm{LP}^{-1}(\phi) = \mu + \left(\tfrac{3}{8}e^2 + \tfrac{3}{16}e^4 + \ldots\right)\sin 2\mu + \left(\tfrac{21}{256}e^4 + \ldots\right)\sin 4\mu + \ldots$$

Rectifying latitude, $\mu = \dfrac{\pi/2}{\mathrm{E}(e)}\phi$; E($e$) is complete elliptic integral of 2nd kind, $\dfrac{\mathrm{E}(e)}{\pi/2} = 1 - \tfrac{1}{4}e^2 - \tfrac{3}{64}e^4 - \ldots$

Conclusions

- Flawed procedures for advancing CoP's have impeded finding solution to the running fix on the sphere
- Approached correctly the solution can be found simply and efficiently
- Running fix on the ellipsoid is modest extension from sphere
- Conclusions give here available in Stuart, R. G. (2017). The Running Fix on an Ellipsoid. *The Journal of Navigation*, **70**, 440–445. http://www.cambridge.org/core/services/aop-cambridge-core/content/view/9AA24FEF390E1CEC5F26962ADF08633D/S0373463316000837a.pdf/running_fix_on_an_ellipsoid.pdf
- More real world solution given in Kaplan, G. H.(1996) A Navigation Solution Involving Changes to Course and Speed, *NAVIGATION: Journal of the Institute of Navigation*. **43**, 469-482.