

The Direct Fix of Latitude and Longitude from Two Observed Altitudes

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ABSTRACT

This work presents a direct method for obtaining the latitude and longitude of an observer from the observed altitudes of two celestial bodies. No assumed position or dead-reckoned position or plotting is required. Starting with the Greenwich hour angles, declinations, and observed altitudes of each pair, the latitude and longitude of the two points from which the observations must have been made are directly computed. The algorithm is presented in the paper, along with its derivation.

Two different, inexpensive, programmable pocket electronic calculators were programmed to execute the algorithm, and they do it in under 30 s. The algorithm was also programmed to run on a personal computer to examine the effect of the precision of the calculations on the error in the results. The findings show that the use of eight decimal places in the trigonometric computations provides acceptable results.

INTRODUCTION

Background

The observation of the altitude, h , of a celestial body places the observer on a small circle centered at the geographical position (GP) of the body. This is called a constant altitude circle, or circle of equal altitudes [1]. An example of a constant altitude circle is shown in Figure 1. The distance from the GP, measured along a great circle that passes through it, to the constant altitude circle is equal to the zenith distance of the body, or to its equal, the complement of the altitude— $(90 - h)$ degrees. The longitude at each point on the constant altitude circle can be found from the latitude. By this means the points on the constant altitude circle may be transferred to a chart to plot line of position.

The observation of the altitude of a second body similarly places the observer on another constant altitude circle that is centered at the GP of the second body. An example of two constant altitude circles is shown in Figure 2. The intersections of

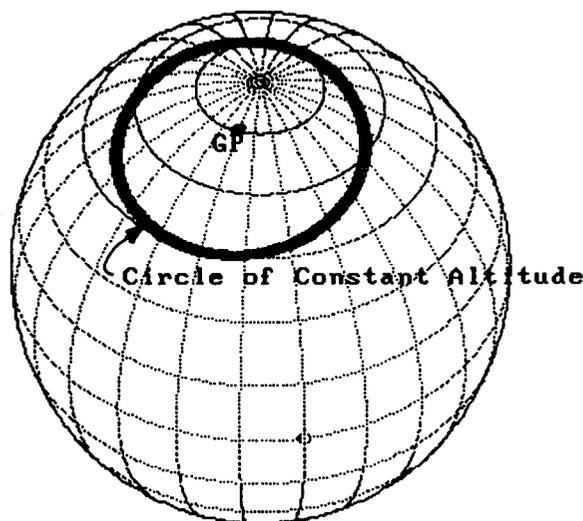


Fig. 1—Circle of Constant Altitude

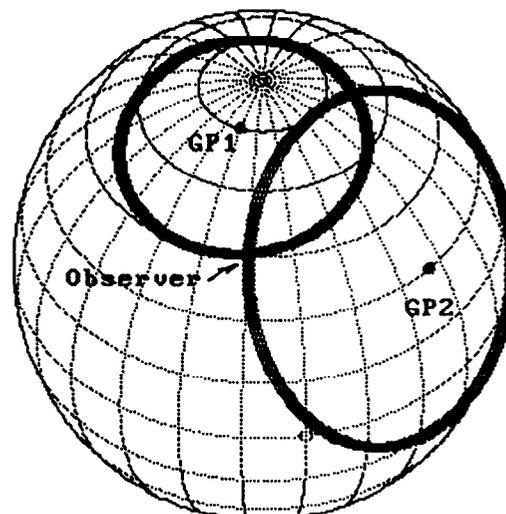


Fig. 2—Two Circles of Constant Altitude Intersecting for a Fix

these two constant altitude circles are the two possible points from which the pair of observed altitudes could have been measured. One of the two points is the location of the observer, a fix, while the other is extraneous. Identification of the correct intersection can be effected by several means, some of which are the consistency of the azimuths of the bodies at each intersection with the bearings of the bodies at the time of the observations; the consistency of the fix with the dead reckoning (DR) at the time of the sights; and, if available, the consistency of the candidate fix with the altitude observed for a third body.

An algorithm that produces the latitudes and longitudes of the two points of intersection of the pair of constant altitude circles is presented in a later section. It provides a fix for the observer directly from the declinations, Greenwich hour angles (GHAs), and observed altitudes of the two bodies. A heuristic proof of the existence of a solution is the fact that a solution can be found by construction on a globe. A compass could be used to scribe the two constant altitude circles on a globe, and the coordinates of the intersections could then be found by measurement. But because of the limitation of scale—being, for example, about 17 mi/mm for a 16 in diameter globe, which would result in a 0.5 mm pencil line being about 8 mi wide—this procedure is of little, if any, practical interest.

Spherical Trigonometric Solution

At first a geocentric, rectangular coordinate system solution based on a “rho-rho-rho” geometry was sought: each intersection of the constant altitude circles was at a known distance from three points—the geocenter and the two GPs. This is similar to the matrix approach proposed in [2], the rotational method of [3], and the solid geometry technique of [4]. But that solution was set aside in favor of a spherical trigonometric approach. This was later found to be identical in principle to an approach proposed in [5], but with considerable simplification of the calculations, and with the law of cosines used instead of the law of sines and the simultaneity requirement removed. The method of [5] uses the available data to provide the parallactic angle in the navigational triangle for either of the two observed bodies. These triangles are then known “side-angle-side,” and either can be solved for the latitude and meridian angle of the observer. Reference [5] proposed simultaneous sights in order to tabulate selected star pairs according to their separation in right ascension. This is not a requirement here.

The compact, low-power, lightweight, low-cost, fast, accurate, and reasonably rugged electronic

calculator makes this direct fix approach practical. In the time of [5] (1949), the required calculations would have been carried out by hand, probably using log-trig tables. Therefore, the law of sines, $\arcsin[\sin(a)\sin(b)/\sin(c)]$, would be easier to calculate than the law of cosines, $\arccos[\cos(d)\cos(e)\cos(f) + \sin(e)\sin(f)]$, because the addition in the latter would necessitate antilog operations out of the tables to perform the addition. But the range of the solution with the law of sines is limited to 90 deg, and so quadrantal difficulties may arise. The law of cosines has a range of zero to 180 deg, and its calculation with a modern calculator is easy, fast, and accurate.

FORMULATION OF THE DIRECT FIX METHOD

For convenience in using the direct fix, the celestial body that is west of the other one of the pair by less than 180 deg is designated number one, at GP1, with decl1, GHA1, and h1; similarly for the other body, at GP2, with decl2, GHA2, and h2. The special case in which the two bodies have the same GHA is dealt with separately. The convention that west longitudes are positive, east are negative, and north declinations and latitudes are positive, south are negative is used.

The essence of this direct fix is to use the data available to compute the parallactic angle in the navigational triangle(s) that use body number 1. This makes the navigational triangle known “side-angle-side” and permits computation of the latitude of the observer using the law of cosines. The navigational triangle is then known “side-side-side,” and its meridian angle can be computed, again using the law of cosines (rearranged into the “time-sight” form), to produce the longitude. To solve for the parallactic angle referred to, two spherical triangles, the *polar triangle* and the *zenith triangle*, are introduced.

The Polar Triangle

This triangle is formed by the arcs of two great circles, meridians that join each of the two GPs to the North Pole, and with the arc of the great circle that joins the two GPs. Figure 3 shows the polar triangle for an example case. The choice of the North Pole is arbitrary, and is based upon convenience. The sides of this pseudonavigational triangle are the codeclinations of the bodies; the meridian angle is the smaller angle between their hour circles (or the smaller difference in their GHAs). The arc joining the two GPs is similar to the coaltitude of a navigational triangle and so is called the pseudocoaltitude. The pseudoaltitude is designated by h12.

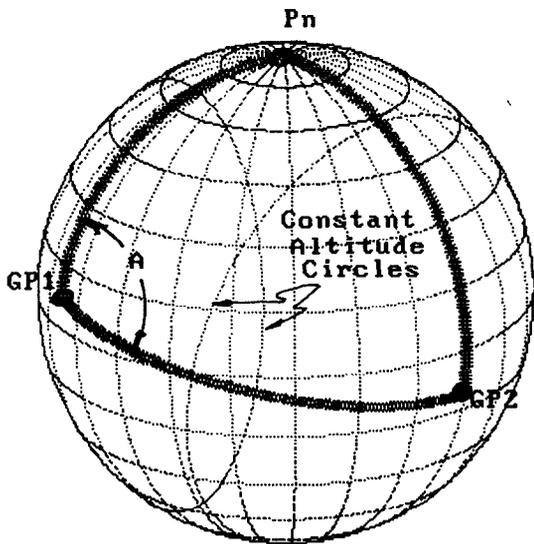


Fig. 3—The Polar Triangle

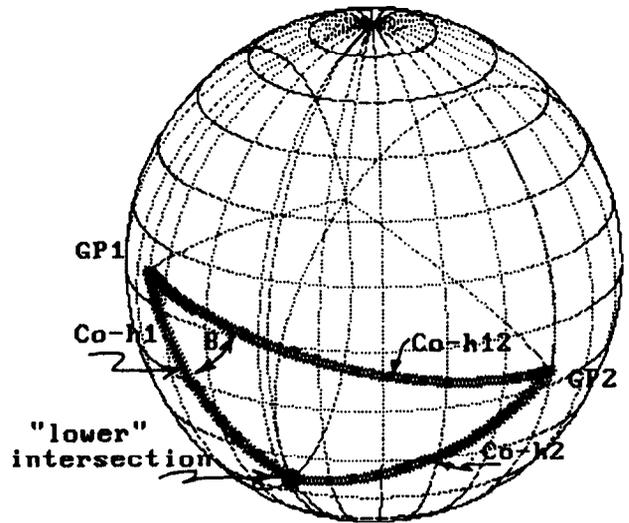


Fig. 5—Zenith Triangle for "Lower" Intersection

The Zenith Triangles

A zenith triangle is formed by the three points GP1, GP2, and one of the intersections of the constant altitude circles. The three sides of these triangles are known (after h12 is found). They are the complements of the altitudes h1, h2 for bodies 1 and 2, respectively, and the complement of the pseudoaltitude, h12. The angle at GP1 is the same for the two zenith triangles. The zenith triangles for an example case are illustrated in Figures 4 and 5.

Latitude and Meridian Angle

It can now be discerned, from Figure 6, that the parallactic angle, P1, in the navigational triangle for the upper intersection is given by angle A at GP1 for the polar triangle, minus angle B at GP1 for the zenith triangle, that is, $P1 = A - B$. From Figure 7 for the lower intersection, the parallactic

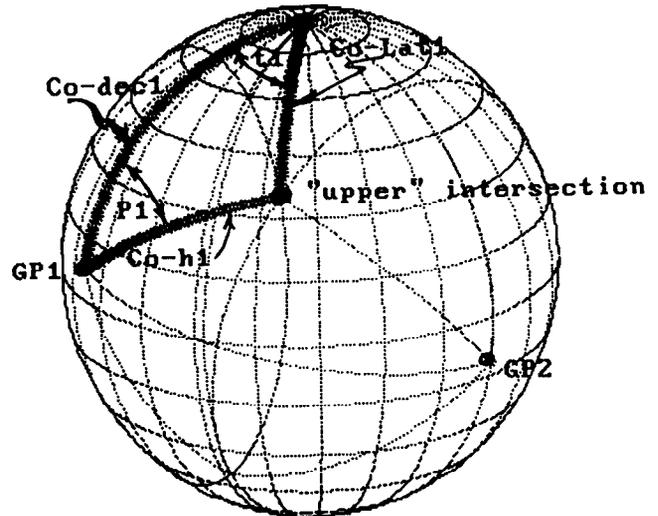


Fig. 6—Navigational Triangle for "Upper" Intersection

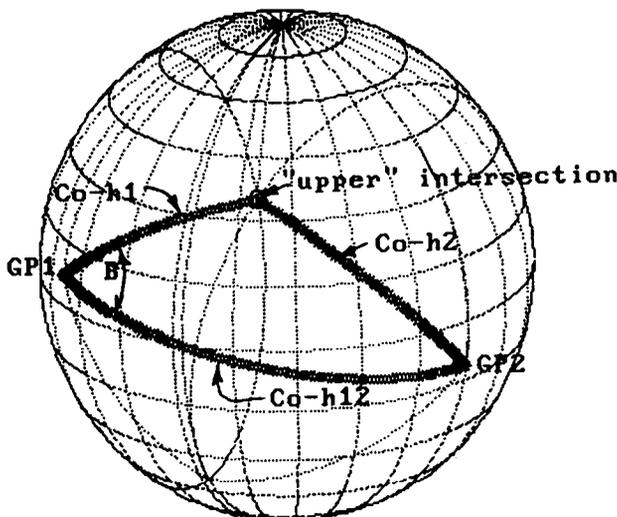


Fig. 4—Zenith Triangle for "Upper" Intersection

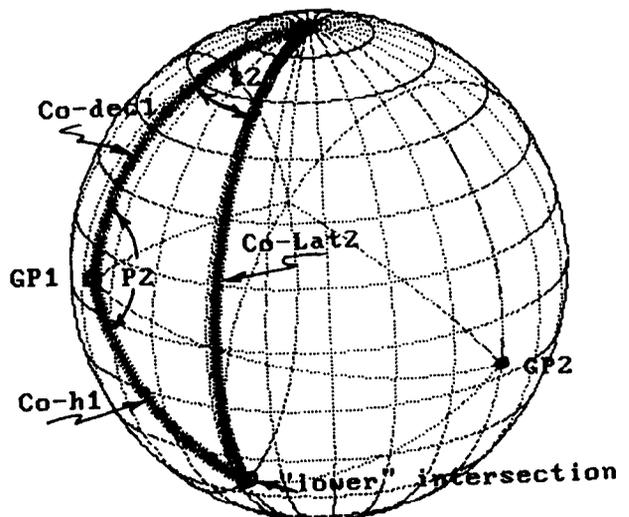


Fig. 7—Navigational Triangle for "Lower" Intersection

angle, P2, in the navigational triangle is found from the sum $P2 = A + B$. The two navigational triangles are now known "side-angle-side," and the latitudes of the intersections can be found from the law of cosines formula for altitude:

$$\sin(L_{1,2}) = \cos(P_{1,2})\cos(d1)\cos(h1) - \sin(d1)\sin(h1) \quad (1)$$

With the latitudes found, the navigational triangles are now known "side-side-side." The meridian angle is opposite the coaltitude in these triangles and is given by the law of cosines, rearranged:

$$\cos(t_{1,2}) = [\sin(h1) - \sin(L_{1,2})\sin(d1)] / [\cos(L_{1,2})\cos(d1)] \quad (2)$$

THE DIRECT FIX PROCEDURE

List of Symbols and their Significance

Input Data

GHA1 is Greenwich hour angle of body number 1

GHA2 is Greenwich hour angle of body number 2

d1 is declination of body number 1

d2 is declination of body number 2

h1 is measured altitude of body number 1

h2 is measured altitude of body number 2

Intermediate Calculations

t12 is meridian angle (in the polar triangle)

h12 is pseudoaltitude (altitude of the body at GP1 from GP2)

A is angle at GP1 for the polar triangle

B is angle at GP1 for the zenith triangle

P1 is parallactic angle for the "upper" navigational triangle using GP1

P2 is parallactic angle for the "lower" navigational triangle using GP1

t1 is meridian angle for the "upper" navigational triangle using GP1

t2 is meridian angle for the "lower" navigational triangle using GP1

Results

L1 is latitude of the "upper" intersection of the constant altitude circles

L2 is latitude of the "lower" intersection of the constant altitude circles

Lo1 is longitude for the "upper" intersection of the constant altitude circles

Lo2 is longitude for the "lower" intersection of the constant altitude circles

Algorithm for the Direct Fix

Subroutines

$$\begin{aligned} \text{alt}(a,b,c) = & \arcsin[\cos(a)\cos(b)\cos(c) \\ & + \sin(b)\sin(c)] \end{aligned} \quad (1a)$$

$h = \text{alt}(a,b,c)$ is the complement of the side opposite the angle a in the spherical triangle, with angle a between the sides that are the complement of b and the complement of c .

$$\begin{aligned} \text{azi}(x,y,z) = & \arccos[(\sin(x) \\ & - \sin(y)\sin(z))/(\cos(y)\cos(z))] \end{aligned} \quad (2a)$$

$Z = \text{azi}(x,y,z)$ is the angle opposite the side that is the complement of x in the spherical triangle, with sides that are the complements of x , y , and z .

Procedure

1. $t12 = \text{GHA2} - \text{GHA1}$ (Note: It is not necessary that $t12$ be positive or less than 180 deg.)
2. $h12 = \text{alt}(t12,d1,d2)$ (Note: $h12$ may be negative if body 1 is below the horizon at GP2.)
3. $A = \text{azi}(d2,d1,h12)$
4. $B = \text{azi}(h2,h1,h12)$
5. $P1 = A - B$ $P2 = A + B$
6. $L1 = \text{alt}(P1,d1,h1)$ $L2 = \text{alt}(P2,d1,h1)$
7. $t1 = \text{azi}(h1,d1,L1)$ $t2 = \text{azi}(h1,d1,L2)$
8. Combine the meridian angles with GHA1 to form the longitudes.
9. The simple algorithm:
If body number 1 is sighted to the west, set $Lo1 = \text{GHA1} - t1$ and $Lo2 = \text{GHA1} - t2$.
If body number 1 is sighted to the east, set $Lo1 = \text{GHA1} + t1$ and $Lo2 = \text{GHA1} + t2$.

The simple algorithm is to be used when the bearing of body number 1 is reliably known to be to the east or west of the observer. This will always then yield the correct longitude for that one of the two intersections of the constant altitude circles from which the observer took the sights. However, the bearing of body number 1 from the unoccupied intersection may not be the same as that from the occupied one. In that case, the computation of $Lo2$ with the simple algorithm will be incorrect. This would be of significance only if it interfered with the correct resolution of the ambiguity of position, that is, selection of the fix and elimination of the extraneous solution.

If this is suspected, or if the bearing of body number 1 is too close to the observer's meridian to identify it reliably as east or west, the full algorithm (definition to follow) should be used.

10. The full algorithm:
Compute the altitude of body number 2 from the four positions:
Position number 1: $Lo = \text{GHA1} - t1$,
Lat = L1

Position number 2: $Lo = GHA1 + t1$,
 Lat = L1
 Position number 3: $Lo = GHA1 - 2$,
 Lat = L2
 Position number 4: $Lo = GHA2 + t2$,
 Lat = L2

Using:

$h21m = \text{alt}(t12 - t1, d2, L1)$

$h21p = \text{alt}(t12 + t1, d2, L1)$

$h22m = \text{alt}(t12 - t2, d2, L2)$

$h22p = \text{alt}(t12 + t2, d2, L2)$.

Compare $h21m$ and $h21p$ with $h2$ for the "upper" intersection

If $|h2 - h21m|$ is smaller than $|h2 - h21p|$, use the longitude of position 1; else use the longitude of position 2.

Compare $h22m$ and $h22p$ with $h2$ for the "lower" intersection

If $|h2 - h22m|$ is smaller than $|h2 - h22p|$, use the longitude of position 3; else use the longitude of position 4.

Pocket Calculator Programs

The simple algorithm has been programmed into the Hewlett-Packard model 15C and the Radio Shack model EC-4026 programmable pocket calculators. The running time, exclusive of data entry, is under 30 s for both calculators. The input data (the 2 GHAs, declinations, and altitudes) are first converted, using the calculator manually, from degrees, minutes, and tenths of minutes to decimal degrees, and are stored in the six reserved locations in the calculator. The code for the EC-4026 program is presented in Appendix A.

EXAMPLES

The algorithm is applied to two artificial cases and one case of real observations. The altitudes cited are the "observed" altitudes, corrected for index error, dip, and refraction. Angles given in degrees, minutes, and tenths of a minute were converted first to decimal degrees ($dd = \text{deg} + \text{arcmin}/60$).

Artificial Cases

- For the values used in Figure 2:
 $dec1 = 75^\circ$, $GHA1 = 30^\circ$, $h1 = 60^\circ$
 $dec2 = 30^\circ$, $GHA2 = 320^\circ$, $h2 = 45^\circ$
 $L1 = 68.52709349^\circ$ N, $Lo1 = 80.29117843^\circ$ E
 $L2 = 45.73917878^\circ$ N, $Lo2 = 14.72877829^\circ$ W
- Consider the direct reduction of sights taken of Betelgeuse and Spica from latitude 35° north and longitude 20° east at 0600 UT on October 28, 1993. Using the Nautical Almanac [6], the GHA of Aries ($126^\circ 35.9'$) and the following data are found and/or calculated:

	Betelgeuse	Spica
SHA	$271^\circ 17.0'$	$158^\circ 47.1'$
GHA	$37^\circ 52.9'$	$285^\circ 23.0'$
LHA	$57^\circ 52.9'$	$305^\circ 23.0'$
declination	$7^\circ 24.4'$ N	$11^\circ 07.7'$ S
altitude	30.38611048°	20.77519091°
azimuth	256.81178600°	121.17412990°

Application of the direct algorithm with these GHA, declination, and altitude values leads to:

t12	112.49833333°		
h12	-23.40384988°		
A	98.96405910°		
B	45.42538149°		
P1	53.53867661°	P2	144.38944159°
L1	35.00000000° N	L2	39.06928279° S
t1	57.88166666°	t2	40.31278930°
Lo1	20.00000000° E	Lo2	2.43112263° E

The navigational triangle for the "upper" intersection is presented in Figure 6. The location of the extraneous solution at L2, Lo2 is shown in Figure 7, along with the navigational triangle for the extraneous location. The negative value for $h12$ is of no concern. It reflects the fact that body number 1 is below the horizon from the GP of body number 2 in this example.

Real Observation

Sight data taken on February 4, 1995, from a DR position of $24^\circ 32.8'$ N, $81^\circ 47.8'$ W comprises:

Body	Name	GMT	GHA	Dec	Altitude
1	Spica	11-29-15	$105^\circ 14.1'$	$11^\circ 08.2'$ S	$47^\circ 33.8'$
2	Venus	11-40-31	$39^\circ 43.5'$	$20^\circ 47.7'$ S	$28^\circ 54.8'$

Entering the GHAs, declinations, and altitudes into the Radio Shack Model EC-4026 calculator and executing the program (file 4, DFIX, Appendix A) leads to a fix at $24^\circ 35.6'$ N, $81^\circ 46.4'$ W (and the spurious solution at $53^\circ 28.4'$ S, $101^\circ 36.7'$ W.) This is to be compared with the fix obtained by the altitude-intercept method of Marcq St-Hilaire: $24^\circ 35.6'$ N, $81^\circ 46.4'$ W, the same result to the nearest tenth of a minute. The sight was reduced by the law of cosines, giving altitudes and azimuths of the two bodies from the DR position.

SPECIAL CASE

In the event that the two observed celestial bodies have the same GHA, the polar triangle collapses to a straight line, and a slightly different algorithm is employed. The convention now is that body number 1 is selected as the one that has the more northerly declination. The complement of the angular separation of the two bodies is then formed: $Dd = 90 - (d1 - d2)$, using the convention that southerly declinations are

negative. The fix and the extraneous solution have the same latitude and are equally distant in longitude from the GPs. The parallactic angle is given by $P = 180^\circ - \text{azi}(h_2, Dd, h_1)$, and is the same for both zenith triangles. The latitude is then given by $L = \text{alt}(P, d_1, h_1)$, and the meridian angle is found from $t = \text{azi}(h_1, d_1, L)$. The two longitudes are found by going east and west from the GHA of the bodies by t . An example is given in Appendix B.

ACCURACY ANALYSIS

The organization of this method for a direct fix into subroutines that are repeatedly called to ultimately yield the latitude and longitude of the intersections of the two constant altitude circles masks the extensiveness of the underlying trigonometric and arithmetic computations. To compute the latitude and meridian angle of one of the intersections requires the evaluation of 14 trig functions (sine, cosine, or their inverses) and 14 arithmetic operations (multiplication or division). The effects of calculation with finite word size, truncation errors, and inaccuracies in trigonometric evaluations are not obvious. To determine these effects, the following considerations and computer experiment were applied. The objective was to determine the number of decimal places required so that the resulting error in the direct fix due to word size would be insignificant with respect to 0.1 arcmin.

The number of decimal places that are carried in computation can be controlled by scaling and truncating the values of inputs and outputs from functions, such as the sine or arccosine. This was done for the case of example number 2 by multiplying each value in the computation by 10^n , where n is the number of decimal places desired; dropping any fractional part; and then dividing by 10^n . This was executed, with n ranging from 4 to 12, on a Pentium 90 processor (fault-free), with the result shown in Figure 8. For each value of n used, the tens-logarithm of the magnitude of the difference between the true latitude and the latitude computed by the direct fix is plotted; similarly for the errors in longitude and the root-sum-square (RSS) error. It can be seen that with 7 decimal places, the errors are of the order of 0.06 arcmin. This amounts to about 0.06 nmi and would hardly be noticeable in the results. However, this accuracy analysis is only cursory, and to be conservative, 8 decimal places, with errors of about 0.002 arcmin, is considered acceptable. A typical high-quality pocket calculator (such as the Hewlett-Packard 15C) carries, in effect, 12 decimal places, and should prove quite adequate for the direct fix. Bear in mind that

these accuracy results are for the cascade of the seven applications of the law of cosines, that is, for the entire direct fix method.

MOTION OF THE OBSERVER

The development of the direct fix given above is robust and exact for observations taken from the same place. The observations need not be simultaneous (although they might be with observers side by side, or with a double sextant [5]), but their times must be registered accurately so that the GPs of the bodies at the times of their altitude measurement can be found in the almanac. The observations need not even be taken on the same day. Typically, two sights could be 1 or 2 min apart on two different bodies. The two sights could also be on one body, such as the sun or the moon, separated in time by an amount that results in the two constant altitudes circles crossing sufficiently close to 90 deg for acceptable geometric dilution of accuracy ("cut"). The term "simultaneous" observation was defined in [5] and cited in [7]. The objective of [5] was to tabulate star pairs according to their difference in RA for a tabular solution of the proposed method, the difference in RAs being, for simultaneous sights, the same as the difference in GHAs (the value of t_{12} , the meridian angle in the polar triangle). If a single observer is on a platform that is in motion, and the two observations are sequential in time, and if the later observation is made from a position that is not on the first constant altitude circle, then neither intersection of the two constant altitude circles can coincide with the location of the observer at either time. Techniques for compensating for this effect are discussed in [7–10]. They are all approximations, which can be quite adequate. Reference [7] points out that the motion effect can be ignored in the case of sailing craft, presumably because of the small change in location of the observer between the two sights.

Retirement and Advancement of Constant Altitude Circles

For an exact direct fix that is based on the intersections of two constant altitude circles, with displacement of the observer between sights, each point on the later-time constant altitude circle must be retired to reflect its location at the time of the earlier sight. If the observer's platform can be assumed to be traveling on a rhumb line, each point on the later-time constant altitude circle must be moved, along the loxidromic spiral that corresponds to the rhumb line of motion through that point, by an amount equal to the distance traveled between the observations. Or similarly, each point on the earlier-time constant altitude

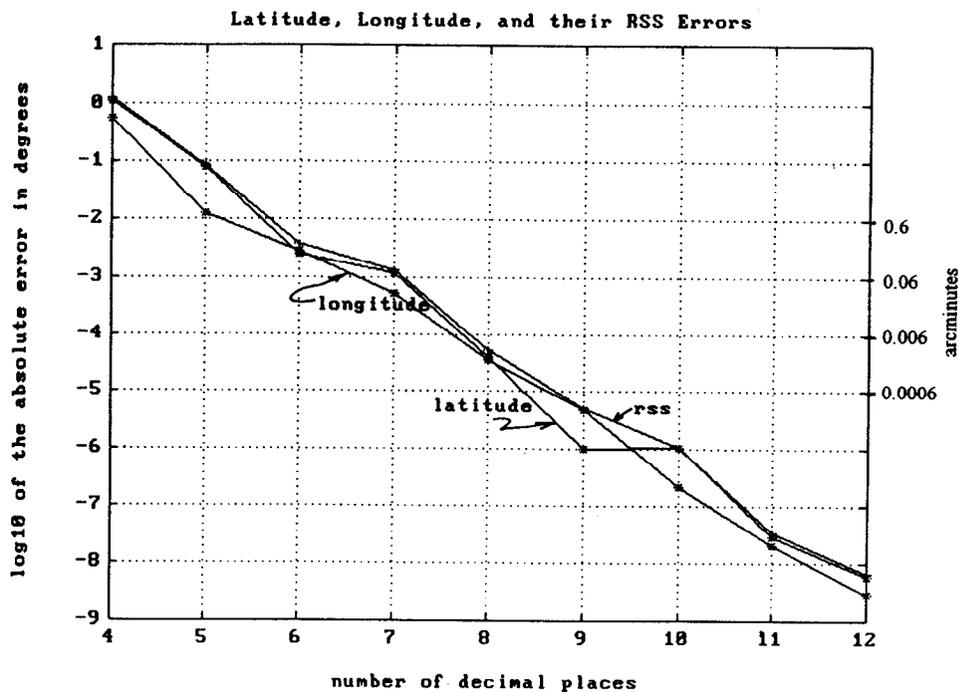


Fig. 8—Error Versus Number of Decimal Places Used in the Calculations

circle must be relocated along a loxidromic spiral to the position it would have at the time of the later observation. This retirement or advancement then produces a new locus for the observer. This locus can readily be shown not to be exactly a circle by considering its generator (the original constant altitude circle) to pass near a pole. The problem of obtaining a fix by solving for the location of the intersections of two constant altitude circles now becomes one of solving for the intersection of, say, the later-time constant altitude circle and the locus of points formed by advancing the points on the earlier-time constant altitude circle. This might be done by searching for the point(s) on the earlier constant altitude circle which when extrapolated to their later position lie on the later constant altitude circle. It is not clear that this has an analytic solution, although approximate methods have been proposed [7–10]. Investigation of this problem beyond this short discussion is considered outside the scope of this paper.

CONCLUSIONS

In summary, it is concluded that by means of an ordinary, programmable pocket calculator, the declinations, Greenwich hour angles, and altitudes from the sights of two celestial bodies can be used to compute directly the latitude and longitude of a stationary observer and of the extraneous intersection of the two constant altitude circles. The selection of the fix and rejection of the extraneous solution can be effected by use of the rough bearings of the bodies when their altitudes

were measured and/or by use of the dead-reckoning position at the time of the sight taking. The advantages of this approach for producing a fix include not having to use a dead-reckoned position or calculate an assumed position; eliminating the need for tables; and eliminating the need for plotting or computing the fix from the results of the altitude-intercept method of Marcq Saint-Hilaire, which is the current sight reduction method of choice.

ACKNOWLEDGMENT

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APPENDIX A

DIRECT FIX CODE FOR PROGRAMMABLE POCKET CALCULATOR, RADIO SHACK MODEL EC-4026

ALTITUDE SUBROUTINE:

FILE number 1: ALT

line 1 Fixm
line 2 $H = \sin^{-1}(\cos A \cos B \cos C + \sin B \sin C)$

AZIMUTH SUBROUTINE:

FILE number 2: AZI

line 1 Fixm
line 2 $Z = (\sin B - \sin C \sin H)$
line 3 $Z = Z / \cos C / \cos H$
line 4 $Z = \cos^{-1} Z$

FILE number 4: DFIX

Line Code	Comment
1 Fixm	use/freeze memory
2 A=U-X	t12 formed and stored in register A
3 B=W	dec1 into register B
4 C=T	dec2 into register C
5 Prog ALT	call alt subroutine, yields h12 = alt(t12,dec1,dec2) in register H
6 B=T	dec2 into register B
7 C=W	dec1 into register C
8 Prog AZI	yields "A" = azi(dec2,dec1,h12), and stores "A" in register Z
9 D=Z	save "A" in register D
10 B=V	h2 into register B
11 C=Y	h1 into register C
12 Prog AZI	yields "B" = azi(h2,h1,h12) into register Z

13 E=Z	save "B" in register E
14 A=D-E	"upper" parallactic angle "P" into register A
15 B=W	dec1 into register B
16 C=Y	h1 into register C
17 Prog ALT	yields Lat1 = alt(P1,d1,h1) in register H
18 F=H, display	saves Lat1 in register F and halts to show Lat1
19 B=Y	h1 into register B
20 C=W	dec1 into register C
21 H=F	Lat1 into register H
22 Prog AZI	yields meridian angle for upper solution in register Z
23 G=Z	save meridian angle in register G
24 I=X-G, display	longitude of upper solution into register I, and halt "lower" parallactic angle "P" into register A
25 A=D+E	dec1 into register B
26 B=W	h1 into register C
27 C=Y	yields Lat2 = alt(P2,d1,h1) in register H
28 Prog ALT	saves Lat2 in register J and halts to show Lat2
29 J=H, display	h1 into register B
30 B=Y	dec1 into register C
31 C=W	yields meridian angle for lower solution in register Z
32 Prog AZI	save meridian angle in register K
33 K=Z	longitude of lower solution in register L, and halt
34 L=X-K, display	

INPUT REGISTER ALLOCATIONS:

T: Declination of body #2, dec2
U: gha of body #2, gha2
V: altitude of body #2, h2
X: gha of body #1, gha1
Y: altitude of body #1, h1
W: declination of body #1, dec1

RESULTS REGISTER ALLOCATIONS:

H: Result of altitude subroutine, ALT(a,b,c)
Z: Result of azimuth subroutine, AZI(x,y,z)
D: save parallactic angle "A"
E: save parallactic angle "B"
F: save Latitude 1, of upper intersection
G: save meridian angle 1, for upper intersection
I: save longitude 1, of upper intersection
J: save Latitude 2, of lower intersection
K: save meridian angle 2, for lower intersection
L: save longitude 2, of lower intersection

APPENDIX B EXAMPLE OF THE SPECIAL CASE

As an example of the special case in which the two bodies have the same GHA, consider an observer in north latitude $2^{\circ} 15.1'$, at east longitude $179^{\circ} 30.9'$, about 400 nmi east of the Gilbert Islands, on March 12, 1993. At morning twilight, Arcturus is observed at 05-30-14 ZT to have an altitude of $51^{\circ} 15.7'$ in the northwest. Soon after, at 05-32-38 ZT, the moon is sighted in the southwest with an altitude of $49^{\circ} 54.7'$. The Greenwich times and dates are found to be 17-30-14 for Arcturus and 17-32-38 for the moon, both on March 11. From the 1993 Nautical Almanac [6] the GHAs of both bodies are found to be $218^{\circ} 05.9'$, with declinations of $19^{\circ} 12.8'$ N for Arcturus, and $17^{\circ} 03.8'$ S for the moon. Knowing the latitude and longitude of the observer permits calculation of the altitude and azimuth of each body:

Body	Name	Altitude	Azimuth Zn
1	Arcturus	$51^{\circ} 15.7'$	298.5°
2	Moon	$49^{\circ} 54.7'$	239.8°

The algorithm for the special case, as given in the main text, is next applied to find the observer's latitude and longitude from the two observed altitudes and the declinations of the the two bodies and their common GHA.

The value of $Dd = 90^{\circ} - (19^{\circ} 12.8' - (-17^{\circ} 03.8'))$ is found to be $53^{\circ} 43.4'$ or 53.72333334° . The parallatic angle is next found from $P = 180^{\circ} - \text{azi}(h2, Dd, h1)$ to be 111.58847130° . The latitude of the intersection of the constant altitude circles is then found from $L = \text{alt}(P, d1, h1)$ to be north 2.25079007° . In degrees and minutes, this amounts to $2^{\circ} 15.04740438'$ or $2^{\circ} 15.0'$. This is in error by 0.1 arcmin compared with the observer's true latitude of $2^{\circ} 15.1'$ (actual error is 0.0526 arcmin before rounding to tenths of minutes).

The meridian angle is next found from $t = \text{azi}(h1, d1, L)$ to be 35.61366409° . Because the bodies were observed to the west, the meridian angle is subtracted from the GHA of the bodies to yield a GHA for the observer of 182.48521060° , which is equivalent to the east longitude of 177.51533080° , or $177^{\circ} 30.9'$ E. This matches the known longitude of the observer.