Celestial Navigation

on the Complex Plane

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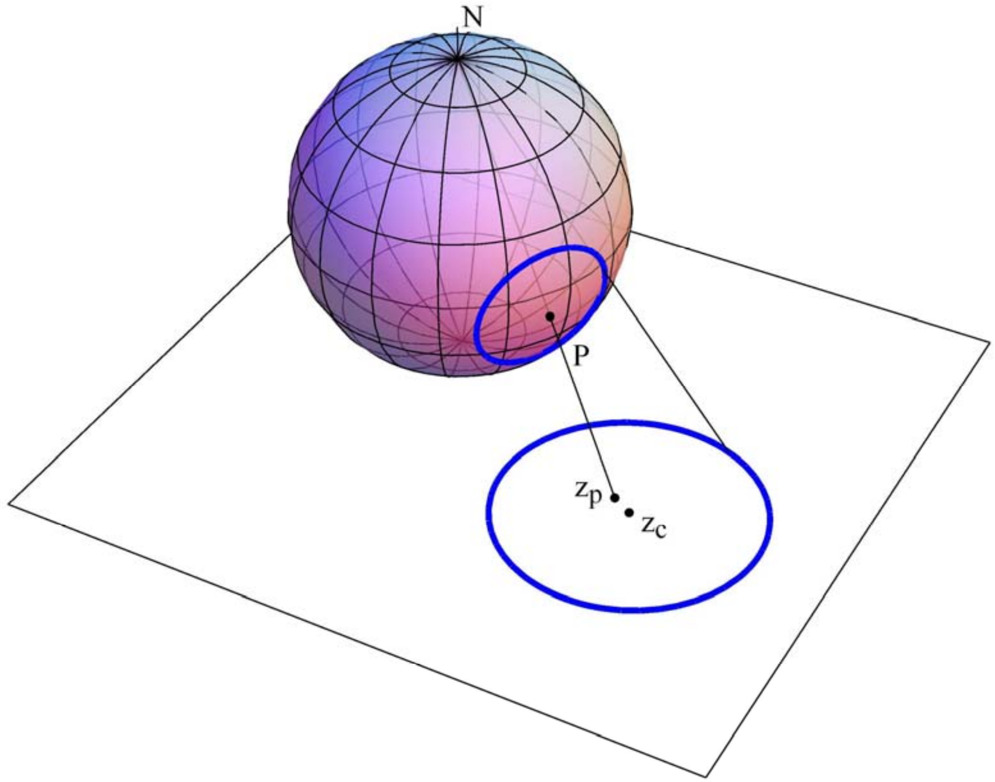
Stereographic Projection



A straight line drawn from *N* projects *P* on the sphere onto z_p on the plane

- Mapping is conformal (angle preserving)
- Circles on the sphere map to circles on the plane
- z_p is generally not the center of the circle on plane

Double Altitude Sight Example

On March 7th,1880 two measurements are made of the Sun's altitude roughly 4 hours apart.

At the first observation

$$\mathrm{GHA}_1 = 0^{\mathrm{h}}59^{\mathrm{m}}59^{\mathrm{s}}.10 \qquad \delta_1 = -4°59'19''.9 \qquad \mathrm{ZD}_1 = 40°00'00''.0$$

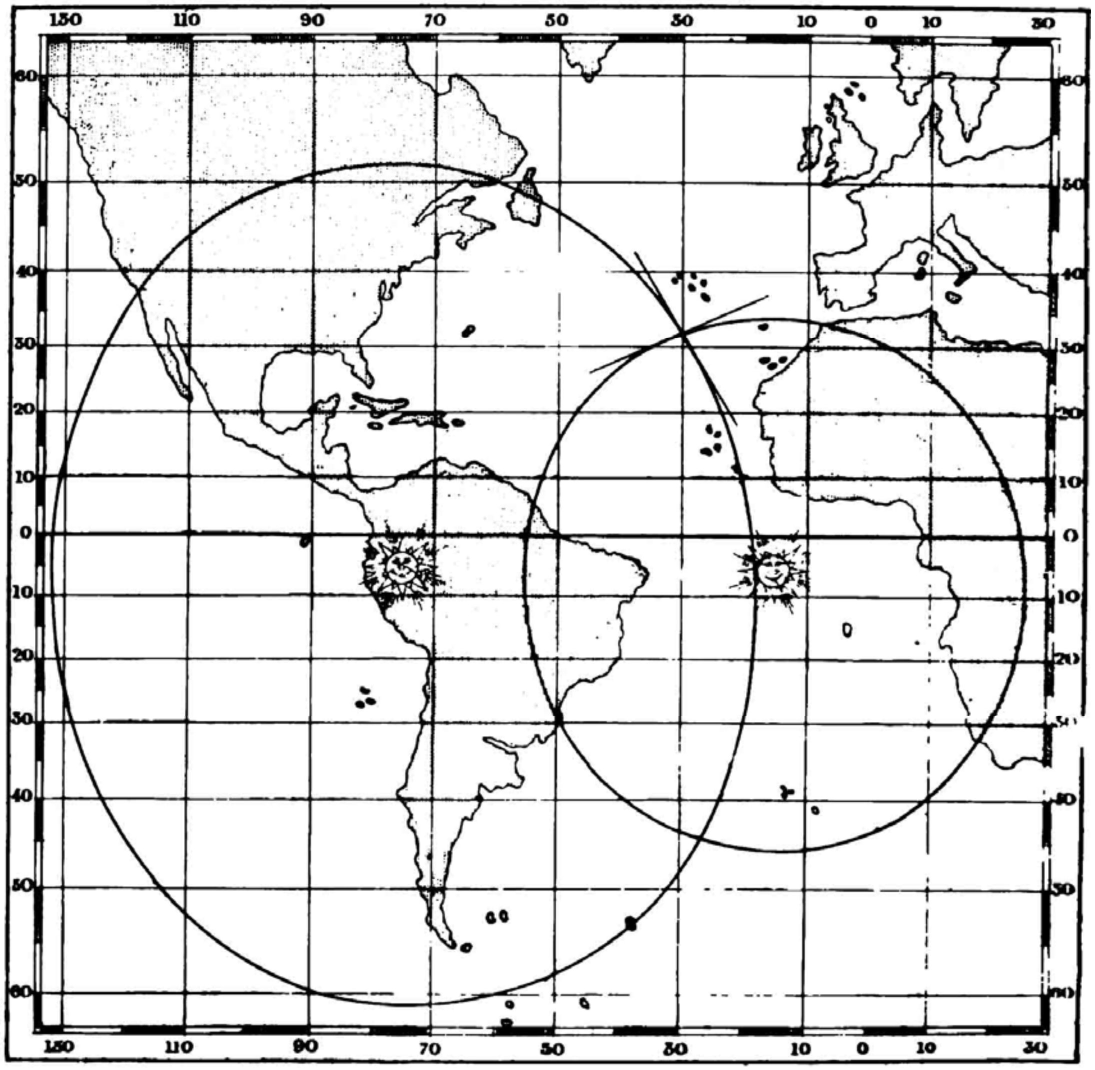
and at the second observation

$$\mathrm{GHA}_2 = 4^{\mathrm{h}}59^{\mathrm{m}}58^{\mathrm{s}}.97 \qquad \delta_2 = -4°55'26''.0 \qquad \mathrm{ZD}_2 = 56°43'15''$$

Lecky, S. T. S., *"Wrinkles" in Practical Navigation*, George Philip & Son, Liverpool, 1886; http://books.google.com/books?id=dmbOAAAAMAAJ

- Each observation defines a Circle of Position (COP)
- Observer's position is at one of the intersections
- For a running fix the radius of one of the circles is adjusted

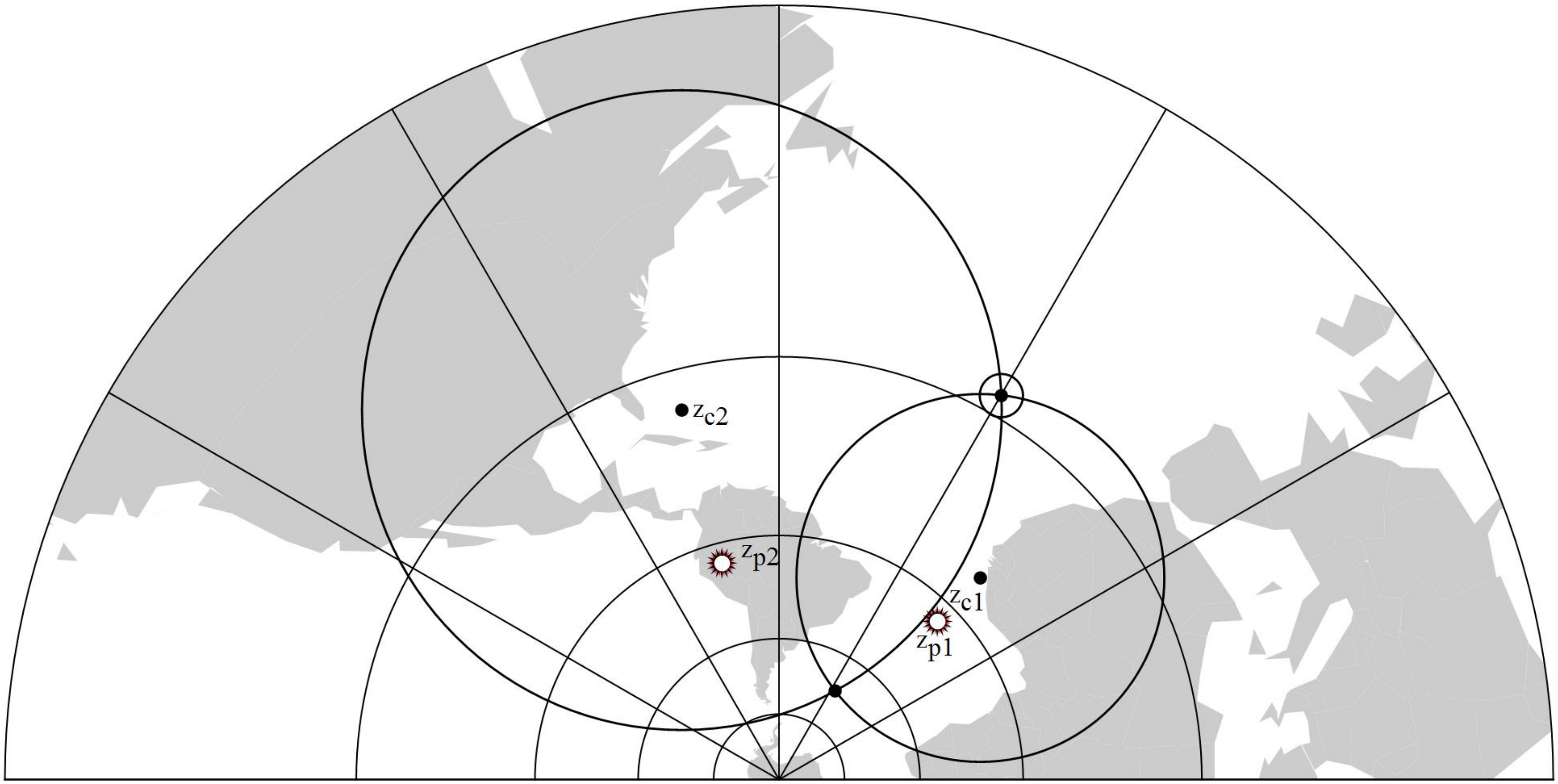
CHARTLET No. 3.



- Intersection at Latitude 32°23´ N Longitude 30° W
- Under Mercator projection COP’s are not circles

Double Altitude Sight Example

Stereographic Projection



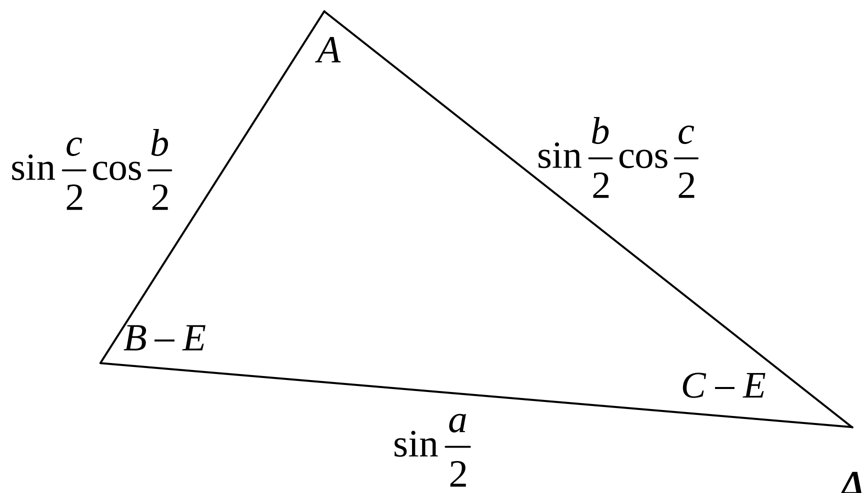
- z_{p1} and z_{p2} are the geographic positions (GP) of the Sun
- Centers of the COP's, z_{c1} and z_{c2}, found graphically from circle vertices and GP's

Observer's position can be located by purely graphical methods!

Stereographic Projection

(Aside)

Stereographic projection of spherical triangles produces auxiliary plane triangles



$$A+B+C-180^\circ = 2E$$

Spherical Excess

Spherical Trigonometry identities are derived by applying standard identities from Plane Trigonometry

Donnay, J. D. H., *Spherical Trigonometry after the Cesàro Method*, Interscience Publishers, Inc., New York, 1945.

- origins in crystallography

Complex Numbers

Complex number, *z*, consists of a real part *x* and imaginary part *y*

$$z = x + iy = r\left(\cos\phi + i\sin\phi\right) = re^{i\phi}$$

where $i = \sqrt{-1}$ and *r*, *φ* are the modulus, argument of *z*

$$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y, \qquad \left|z\right| = r, \qquad \arg\left(z\right) = \phi$$

Complex conjugate of z is

$$\overline{z} = x - iy = r\left(\cos\phi - i\sin\phi\right) = re^{-i\phi}$$

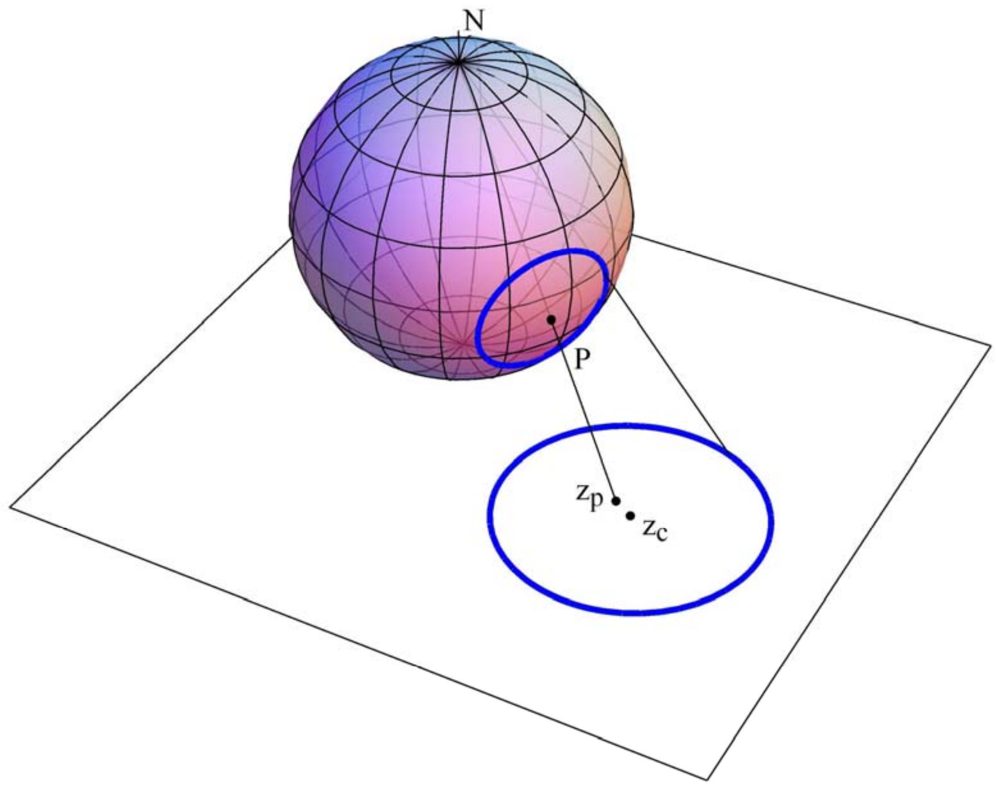
Application to Celestial Navigation uses simple arithmetic of complex numbers

- Built into many scientific calculators
- Native feature of many programming languages; FORTRAN, C++, PERL,…

Functions of complex variables are intimately connected to conformal mapping

- Complex numbers represented as points on a plane or on the Riemann sphere
- Related by stereographic projection

Stereographic Projection of the Globe onto the Complex Plane



Point *P* at latitude *L* and longitude *λ* then $z_p = \tan\left(\frac{\pi}{4} + \frac{L}{2}\right) e^{i\lambda}$ (NP Projection)

or $z_p = \tan\left(\frac{\pi}{4} - \frac{L}{2}\right) e^{-i\lambda}$ (SP Projection)

Two spherical coordinates *L*, *λ* carried in a single complex variable

Properties of Circles on the Complex Plane

Two points z and z_p separated by angular distance θ on the sphere satisfy

$$\tan\frac{\theta}{2} \equiv \rho = \left|\frac{z - z_p}{1 + \bar{z}_p z}\right|$$

The set of points z describes a circle with z_p as its pole

On the complex plane that circle will have center, z_c, and radius, r

$$z_c = \frac{1+\rho^2}{1-\rho^2\left|z_p\right|^2} z_p, \qquad r = \frac{1+\left|z_p\right|^2}{\left|1-\rho^2\left|z_p\right|^2\right|}\rho$$

In the case of a Great Circle $\rho = 1$

$$z_c = \frac{2z_p}{1-\left|z_p\right|^2}, \qquad r = \frac{1+\left|z_p\right|^2}{\left|1-\left|z_p\right|^2\right|}$$

From which it follows $r^2 = \left|z_c\right|^2 + 1$

Intersection of Two Circles on the Complex Plane

Double altitude sight requires finding the intersection of 2 circles on the complex plane

Circle with centers at z_{c1}, z_{c2} and radii r_1, r_2 intersect at

$$z = \frac{1}{2}\left(z_{c1} + z_{c2}\right) + \left(\mu \pm i\nu\right)\left(z_{c2} - z_{c1}\right)$$

where

$$d = \left|z_{c1} - z_{c2}\right|$$

$$\mu = \frac{r_1^2 - r_2^2}{2d^2}$$

$$\nu = \frac{1}{2d^2}\sqrt{4r_1^2 d^2 - \left(d^2 + r_1^2 - r_2^2\right)^2}$$

$$= \frac{1}{2d^2}\sqrt{\left(r_1 + r_2 + d\right)\left(r_1 - r_2 - d\right)\left(-r_1 + r_2 - d\right)\left(r_1 + r_2 - d\right)}$$

Position from Double Altitude Sight of the Sun

Inputs							
	Sun at 1st observation						
	GHA_1	0	h	59	m	59.10	s
	Declination, δ_1	-4	°	59	'	19.9	"
	Zenithal Distance, ZD_1	40	°	00	'	00	"
	Sun at 2nd observation						
	GHA_2	4	h	59	m	58.97	s
	Declination, δ_2	-4	°	55	'	26.0	"
	Zenithal Distance, ZD_2	56	°	42	'	15	"
Calculations							
	z_{p1}	0.885296506243674-0.237152085690398i					
	ρ_1	0.363970					
	z_{p2}	0.237547047555908-0.886271202961888i					
	ρ_2	0.539618					
	z_{c1}	1.12810836339466-0.302196212655319i					
	r_1	0.753556					
	z_{c2}	0.406330463157132-1.51599016736917i					
	r_2	1.316722					
	d	1.412182					
	μ	-0.292317					
	v	0.491536					
	z	1.57483079116482-0.909061138152821i					
		0.381583292836668-0.199501109070274i					
Results							
	Latitude, L	32° 23' 01"					
	Longitude, λ	-29° 59' 44"					
	Latitude, L	-43° 24' 28"					
	Longitude, λ	-27° 36' 06"					

$$z_p = \tan\left(\frac{\pi}{4}+\frac{\delta}{2}\right)e^{-i(\text{GHA})}; \quad \rho = \tan\left(\frac{\text{ZD}}{2}\right)$$

$$z_c = \frac{1+\rho^2}{1-\rho^2\left|z_p\right|^2}z_p; \qquad r = \frac{1+\left|z_p\right|^2}{\left|1-\rho^2\left|z_p\right|^2\right|}\rho$$

$$z = \frac{1}{2}\left(z_{c1}+z_{c2}\right)+\left(\mu \pm i\nu\right)\left(z_{c2}-z_{c1}\right)$$

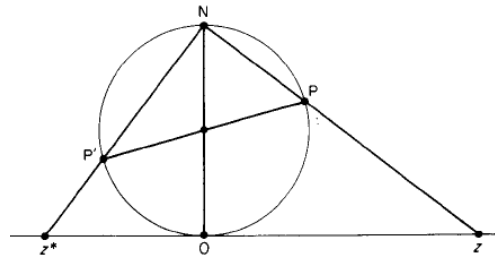
$$L = 2\tan^{-1}\left|z\right| - \frac{\pi}{2}; \quad \lambda = \arg(z)$$

in reference $L = 32°23'\text{N}, \quad \lambda = 30°00'\text{W}$

Rotations on the Complex Plane

If z^* denotes the point diametrically opposite z on the Riemann sphere (antipodal point) then

$$z^* = -1/\bar{z}$$



Can be used to show that the general form of a rotation of the sphere is a bilinear or Möbius transformation

$$\mathrm{T}(z) = \frac{az+b}{-\bar{b}z+\bar{a}}$$

Finding the altitude and azimuth of a celestial body amounts to a rotation of a spherical coordinate system

Equatorial Coordinates (GHA, δ) → Horizontal Coordinates (Z, h)

For Assumed Position (AP) latitude L, longitude λ

$$a = e^{-i\frac{\lambda}{2}}, \qquad b = -\tan\left(\frac{\pi}{4}+\frac{L}{2}\right)e^{i\frac{\lambda}{2}}$$

$$z = \tan\left(\frac{\pi}{4}+\frac{\delta}{2}\right)e^{-i\,(\mathrm{GHA})}; \quad \mathrm{T}(z) \equiv w = \tan\left(\frac{\pi}{4}-\frac{h}{2}\right)e^{iZ}$$

Coefficients a and b depend only on AP – can be used for multiple objects

Altitude and Azimuth

Find the altitude, *h* and azimuth *Z*, of the star Vega (α Lyræ) at 8 o'clock on 24 October, 1874 from assumed Position (AP) latitude *L* = 30°30´N and longitude *λ* = 9°30´W.

Saint Hilaire, A. M., *Revue Maritimes et Coloniale*, **Mar-Aout**, 1875, pp.341-375;
Vanvaerenbergh, M. and Ifland, P., *Line of Position Navigation*, Unlimited Publishing, Bloomington, Indiana, 2003.

	Altitude and Azimuth						
Inputs	**Assumed Position**						
	Latitude, *L*	35	°	30.0	'		
	Longitude, *λ*	-9	°	30.0	'		
	Vega at observation						
	GHA	62	°	16	'	00	"
	Declination, *δ*	38	°	40	'	13	"
Calculations	*a*	0.996565502497761+8.28082075122044E-002i					
	b	-1.93495149010083+0.160782070136608i					
	z	0.968460094085827-1.84204002255684i					
	T(z) ≡ *w* =	0.134118058699435-0.35573213541146i					
Results							
	Altitude, *h*	48° 22' 08"					
	Azimuth, *Z*	290° 39.4'					

$$a = e^{-i\frac{\lambda}{2}}, \qquad b = -\tan\left(\frac{\pi}{4}+\frac{L}{2}\right)e^{i\frac{\lambda}{2}}$$

$$z = \tan\left(\frac{\pi}{4}+\frac{\delta}{2}\right)e^{-i\,(\mathrm{GHA})}$$

$$\mathrm{T}(z) \equiv w = \frac{az+b}{-\bar{b}z+\bar{a}}$$

$$h = \frac{\pi}{2} - 2\tan^{-1}|w|; \quad Z = \arg(w)$$

in reference $h = 48°22'15''$, $\quad Z = 290°40'\,(69°20'\,\mathrm{W})$

Rotations on the Complex Plane

Alternatively can be written

$$\mathrm{T}\left(z_{\mathrm{GP}}\right)=\tan\left(\frac{\pi}{4}-\frac{h}{2}\right)e^{iZ}=e^{-i\lambda}\left(\frac{z_{\mathrm{GP}}-z_{\mathrm{AP}}}{\overline{z}_{\mathrm{AP}}z_{\mathrm{GP}}+1}\right)$$

where z_{AP} is the observer's Assumed Position (AP), $z_{\mathrm{AP}}=\tan\left(\frac{\pi}{4}+\frac{L}{2}\right)e^{i\,\lambda}$

and z_{GP} is the Geographic Position (GP) of the celestial body

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For stereographic projection with complex plane tangent at North Pole

$$a=e^{i\left(\frac{\lambda}{2}-\frac{\pi}{2}\right)},\qquad b=\tan\left(\frac{\pi}{4}+\frac{L}{2}\right)e^{i\left(-\frac{\lambda}{2}+\frac{\pi}{2}\right)}$$

$$z=\tan\left(\frac{\pi}{4}-\frac{\delta}{2}\right)e^{i\,(\mathrm{GHA})};\quad \mathrm{T}\left(z_{\mathrm{GP}}\right)=-e^{i\lambda}\left(\frac{z_{\mathrm{GP}}-z_{\mathrm{AP}}}{\overline{z}_{\mathrm{AP}}z_{\mathrm{GP}}+1}\right)$$

Poles of a Great Circle defined by 2 points

Let z_1 and z_2 be two points on the complex plane

The poles z_p of the great circle passing z_1 and z_2 satisfy

$$\left|\frac{z_p - z_1}{\overline{z}_p z_1 + 1}\right| = \left|\frac{z_p - z_2}{\overline{z}_p z_2 + 1}\right| = 1$$

Solving for z_p gives

$$z_p = i\,\frac{2\,\mathrm{Im}\left(\overline{z}_1 z_2\right) \pm \left|\left(z_1 - z_2\right)\left(1 + \overline{z}_1 z_2\right)\right|}{\left(1 - \left|z_2\right|^2\right)\overline{z}_1 - \left(1 - \left|z_1\right|^2\right)\overline{z}_2}$$

- Determination of sextant arc errors simplified if 2 stars can be found at the same azimuth
- Stars will be at the same azimuth when the pole of their great circles is rising or setting
  - Lord Ellensborough's Method

Sprigge, J. A., Doak, W. F., Hudson, T. C. & Cox, A. S., *Stars and Sextants*, J. D. Potter, London, 1903 http://www.archive.org/details/starssextants00spri

If z_1 and z_2 represent the poles of 2 great circles then the points z_p are their intersections

DISTANCES OF THE STAR PAIRS, Etc.

Star Pair.	Distance.	R.A. and Dec. of Fictitious Star.	
α Ursæ Minoris.			
(*Polaris*) 1h 24m. N. 88° 48′.			
and :—	° ′ ″	h m	°
α Persei (*Mirfak*)	39 25 21	9 20	1 N
α Tauri (*Aldebaran*)	72 51 23	10 31	1 N
α Aurigæ (*Capella*)	43 26 25	11 14	1 N
β Orionis (*Rigel*)	97 38 40	11 9	1 N
γ Orionis (*Bellatrix*)	83 6 58	11 21	1 N
β Tauri (*Nath*)	60 51 27	11 23	1 N
ε Orionis (*Alnilam*)	90 41 34	11 31	1 N
ζ Orionis	91 26 39	11 36	1 N
α Orionis (*Betelgeuse*)	82 7 48	11 51	1 N
β Canis Majoris (*Mirzam*)	107 33 47	12 17	1 N
γ Geminorum (*Alhena*)	73 15 2	12 34	1 N
ε Canis Majoris (*Adara*)	118 40 48	12 52	1 N
δ Canis Majoris	116 7 50	13 2	1 N
α Geminorum (*Castor*)	57 55 50	13 32	1 N
β Geminorum (*Pollux*)	61 49 41	13 42	1 N
α Leonis (*Regulus*)	78 20 8	16 4	1 N
α Ursæ Majoris (*Dubhe*)	28 42 23	17 3	1 N
ε Ursæ Majoris (*Alioth*)	See	page	38.
α Virginis (*Spica*)	See	page	38.
η Ursæ Majoris (*Benetnasch*)	See	page	39.
α Scorpii (*Antares*)	117 4 16	22 17	1 S
α Lyræ (*Vega*)	51 34 46	0 30	1 S
α Aquilæ (*Altair*)	81 16 22	1 45	1 S
α Cygni (*Deneb*)	44 41 33	2 34	1 S
α Piscis Australis (*Fomalhaut*)	119 10 43	4 54	1 S
α Eridani.			
(*Achernar*) 1h 34m. S. 57° 43′.			
and :—			
α Persei (*Mirfak*)	109 20 5	8 32	9 S
α Tauri (*Aldebaran*)	82 29 25	10 2	21 S
α Aurigæ (*Capella*)	112 49 57	9 47	19 S
β Orionis (*Rigel*)	64 19 43	11 26	28 S
γ Orionis (*Bellatrix*)	78 24 8	11 7	27 S
β Tauri (*Nath*)	98 20 9	10 26	23 S
ε Orionis (*Alnilam*)	73 4 15	11 32	28 S
ζ Orionis	72 59 0	11 37	28 S
α Orionis (*Betelgeuse*)	82 53 30	11 32	28 S
β Canis Majoris (*Mirzam*)	64 52 4	13 2	32 S
γ Geminorum (*Alhena*)	95 55 2	11 50	30 S
ε Canis Majoris (*Adara*)	60 48 52	14 15	31 S
δ Canis Majoris	64 10 31	14 9	31 S
α Geminorum (*Castor*)	115 59 0	12 2	29 S
β Geminorum (*Pollux*)	114 16 44	12 26	31 S

Star Pair.	Distance.	R.A. and Dec. of Fictitious Star.	
α Eridani—*continued.*	° ′ ″	h m	°
γ Argûs	55 25 8	16 9	26 S
ε Argûs	47 50 8	5 1	21 N
δ Argûs	53 23 58	17 0	21 S
β Argûs	44 32 7	5 59	14 N
α Leonis (*Regulus*)	119 51 50	15 35	29 S
α Crucis	58 53 11	7 1	5 N
γ Crucis	64 52 56	19 1	7 S
β Crucis	62 39 34	7 6	4 N
α Virginis (*Spica*)	111 33 18	19 18	3 S
β Centauri	See	page	40.
α Scorpii (*Antares*)	88 53 2	21 47	18 N
α Trianguli Australis	49 5 8	8 42	11 S
λ Scorpii	73 27 3	22 11	22 N
θ Scorpii	67 58 32	22 5	21 N
ε Sagittarii	70 29 11	22 55	26 N
α Aquilæ (*Altair*)	95 41 15	2 4	31 N
α Pavonis	40 6 52	22 57	27 N
α Cygni (*Deneb*)	119 32 10	4 22	25 N
α Gruis	32 50 58	1 7	32 N
α Piscis Australis (*Fomalhaut*)	39 6 55	3 38	27 N
α Persei.			
(*Mirfak*) 3h 18m. N. 49° 31′.			
and :—			
α Tauri (*Aldebaran*)	36 20 50	10 52	18 N
α Aurigæ (*Capella*)	19 5 26	15 15	40 N
β Orionis (*Rigel*)	62 49 54	10 57	20 N
γ Orionis (*Bellatrix*)	50 20 17	11 32	25 N
β Tauri (*Nath*)	31 22 45	12 45	33 N
ε Orionis (*Alnilam*)	58 21 21	11 28	24 N
ζ Orionis	59 30 7	11 32	25 N
α Orionis (*Betelgeuse*)	52 49 28	12 7	29 N
β Canis Majoris (*Mirzam*)	78 24 18	11 42	26 N
γ Geminorum (*Alhena*)	51 10 54	13 22	36 N
ε Canis Majoris (*Adara*)	92 1 55	11 44	27 N
δ Canis Majoris	90 58 46	12 0	28 N
α Geminorum (*Castor*)	49 0 24	15 32	39 N
β Geminorum (*Pollux*)	53 19 24	15 34	39 N
γ Argûs	114 58 10	11 52	27 N
α Leonis (*Regulus*)	87 49 26	16 40	37 N
α Ursæ Majoris (*Dubhe*)	56 57 44	7 42	19 S
ε Ursæ Majoris (*Alioth*)	69 39 53	8 10	13 S
η Ursæ Majoris (*Benetnasch*)	78 40 34	8 26	9 S
α Lyræ (*Vega*)	81 45 4	23 14	23 S
α Aquilæ (*Altair*)	97 46 18	1 22	37 S
α Cygni (*Deneb*)	62 41 19	0 9	31 S
α Gruis	118 21 28	6 33	28 S
α Piscis Australis (*Fomalhaut*)	98 58 20	6 15	31 S

Look for the Star with the *smaller* R.A. in bold type.

DISTANCES OF THE STAR PAIRS, Etc.

Star Pair.	Distance.	R.A. and Dec. of Fictitious Star.	
α Tauri.			
(*Aldebaran*) 4h 30m. N. 16° 19′.			
and :—	° ′ ″	h m	°
α Aurigæ (*Capella*)	30 41 44	10 12	13 S
β Orionis (*Rigel*)	26 29 54	10 56	22 N
γ Orionis (*Bellatrix*)	15 45 29	11 54	51 N
β Tauri (*Nath*)	16 45 22	9 34	37 S
ε Orionis (*Alnilam*)	23 8 2	11 26	41 N
ζ Orionis	24 25 50	11 27	42 N
α Orionis (*Betelgeuse*)	21 23 31	12 52	64 N
β Canis Majoris (*Mirzam*)	43 20 54	11 22	37 N
γ Geminorum (*Alhena*)	29 10 23	5 50	73 S
ε Canis Majoris (*Adara*)	57 4 4	11 17	36 N
δ Canis Majoris	56 40 54	11 27	40 N
α Geminorum (*Castor*)	43 11 48	8 48	56 S
β Geminorum (*Pollux*)	45 1 45	8 25	62 S
γ Argûs	79 43 43	11 14	33 N
ε Argûs	88 42 44	11 2	24 N
δ Argûs	88 29 19	11 8	30 N
β Argûs	98 35 22	10 52	18 N
α Leonis (*Regulus*)	80 8 18	18 47	71 N
α Ursæ Majoris (*Dubhe*)	78 43 52	9 56	26 S
α Crucis	117 3 44	11 5	26 N
γ Crucis	119 18 55	11 9	32 N
ε Ursæ Majoris (*Alioth*)	93 56 4	9 58	26 S
η Ursæ Majoris (*Benetnasch*)	104 22 36	9 59	25 S
α Lyræ (*Vega*)	117 52 54	11 2	25 N
α Cygni (*Deneb*)	96 57 34	11 22	37 N
α Gruis	106 39 27	9 27	42 S
α Piscis Australis (*Fomalhaut*)	93 32 15	8 50	55 S
α Aurigæ.			
(*Capella*) 5h 10m. N. 45° 54′.			
and :—			
β Orionis (*Rigel*)	See	page	33.
γ Orionis (*Bellatrix*)	39 42 5	11 24	3 N
β Tauri (*Nath*)	17 29 59	11 32	6 N
ε Orionis (*Alnilam*)	47 24 28	11 32	6 N
ζ Orionis	48 14 48	11 37	7 N
α Orionis (*Betelgeuse*)	39 29 2	11 58	12 N
β Canis Majoris (*Mirzam*)	65 41 20	12 2	13 N
γ Geminorum (*Alhena*)	34 5 6	13 4	25 N
ε Canis Majoris (*Adara*)	78 27 47	12 20	16 N
δ Canis Majoris	76 42 38	12 28	18 N
α Geminorum (*Castor*)	29 59 3	15 50	42 N
β Geminorum (*Pollux*)	34 15 15	15 35	42 N
γ Argûs	100 43 14	12 37	20 N
ε Argûs	112 9 40	12 24	17 N
δ Argûs	109 53 18	12 39	21 N

Star Pair.	Distance.	R.A. and Dec. of Fictitious Star.	
α Aurigæ—*continued.*	° ′ ″	h m	°
α Leonis (*Regulus*)	69 35 50	16 56	44 N
α Ursæ Majoris (*Dubhe*)	49 17 5	9 17	25 S
ε Ursæ Majoris (*Alioth*)	64 9 8	9 26	23 S
η Ursæ Majoris (*Benetnasch*)	74 25 23	9 37	21 S
α Lyræ (*Vega*)	93 19 39	23 59	11 S
α Aquilæ (*Altair*)	115 13 35	1 27	28 S
α Cygni (*Deneb*)	78 10 33	0 56	23 S
α Piscis Australis (*Fomalhaut*)	113 56 16	6 55	41 S
β Orionis.			
(*Rigel*) 5h 10m. S. 8° 19′.			
and :—			
γ Orionis (*Bellatrix*)	14 47 22	11 12	10 S
β Tauri (*Nath*)	36 55 22	11 9	3 S
ε Orionis (*Alnilam*)	8 50 20	11 32	36 S
ζ Orionis	9 2 33	11 40	44 S
α Orionis (*Betelgeuse*)	18 36 20	11 30	32 S
β Canis Majoris (*Mirzam*)	19 13 47	10 17	58 N
γ Geminorum (*Alhena*)	32 4 19	11 32	38 S
ε Canis Majoris (*Adara*)	32 5 19	10 38	46 N
δ Canis Majoris	32 33 29	10 26	52 N
α Geminorum (*Castor*)	52 12 19	11 36	36 S
β Geminorum (*Pollux*)	51 22 54	11 40	42 S
γ Argûs	53 52 58	10 49	36 N
ε Argûs	62 15 37	10 56	25 N
δ Argûs	62 20 19	10 48	32 N
β Argûs	72 7 14	10 56	19 N
α Leonis (*Regulus*)	75 45 36	13 2	72 S
α Ursæ Majoris (*Dubhe*)	95 56 55	11 26	27 S
α Crucis	90 38 49	10 56	26 N
γ Crucis	93 14 40	10 55	31 N
β Crucis	94 16 4	10 52	28 N
ε Ursæ Majoris (*Alioth*)	110 33 39	11 29	32 S
α Virginis (*Spica*)	119 46 5	9 26	71 N
β Centauri	101 51 21	10 56	22 N
α Trianguli Australis	102 38 20	11 8	3 N
α Pavonis	104 12 59	11 22	23 S
α Gruis	95 4 45	11 35	40 S
α Piscis Australis (*Fomalhaut*)	89 35 55	12 2	58 S
γ Orionis.			
(*Bellatrix*) 5h 20m. N. 6° 16′.			
and :—			
β Tauri (*Nath*)	22 15 50	11 20	0
ε Orionis (*Alnilam*)	8 2 32	11 26	18 N
ζ Orionis	9 9 51	11 30	25 N
α Orionis (*Betelgeuse*)	7 31 47	8 42	81 S
β Canis Majoris (*Mirzam*)	28 9 11	11 35	30 N

Look for the Star with the *smaller* R.A. in bold type.

Angular Distance between Stars

Inputs

α Ursae Minoris (Polaris)						
RA, GHA or Longitude	1	h	22	m	33.70	s
Declination or Latitude	88	°	46	'	26.0	"
α Tauri (Aldebaran)						
RA, GHA or Longitude	4	h	30	m	10.90	s
Declination or Latitude	16	°	18	'	30.0	s

B1900 coordinates

Calculations

z_1	87.456969404585+32.9433408647948i
z_2	0.50971383447846+1.23332226643108i
$\lvert(z_1-z_2)/(\bar{z}_2 z_1+1)\rvert$	0.7380191

$$z = \tan\left(\frac{\pi}{4}+\frac{\delta}{2}\right)e^{i\,(\mathrm{RA})}$$

Results

Angular Distance, *d*	72º 51' 22"

$$d = 2\tan^{-1}\left|\frac{z_1 - z_2}{\bar{z}_2 z_1 + 1}\right|$$

Control

Select

Format for 1st point	(●) hhmmss ( ) ddmmss
Format for 2nd point	(●) hhmmss ( ) ddmmss

in reference $d = 72°51'23''$

Lord Ellenborough's Method

Inputs	**α Ursae Minoris (Polaris)**							
	Right Ascension, α_1	1	h	24	m	00.00	s	
	Declination, δ_1	88	°	48	'	00.0	"	
	α Tauri (Aldebaran)							
	Right Ascension, α_2	4	h	30	m	00.00	s	
	Declination, δ_2	16	°	19	'	00.0	"	
Calculations	z_1	89.1471049562414+34.2203674202285i						
	z_2	0.510768690672111+1.23310470025615i						
	$2\,\mathrm{Im}(z_1\ \bar{z}_2)$	184.8980437						
	$\lvert(z_1-z_2)(1+z_1\ \bar{z}_2)\rvert$	12118.97404						
	$(1-\lvert z_2\rvert^2)\ \bar{z}_1 - (1-\lvert z_1\rvert^2)\ \bar{z}_2$	4587.13828236729-11215.7704498881i						
	z_{p1}	-0.939810543931511+0.384373141684945i						
	z_{p2}	0.91156429268045-0.372820706564592i						
Results	**Fictitious Stars Positions**							
	Right Ascension, α_1	10h 31m 01s						
	Declination, δ_1	0º 52' 27"						
	Right Ascension, α_2	22h 31m 01s						
	Declination, δ_2	-0º 52' 27"						

$$z = \tan\left(\frac{\pi}{4}+\frac{\delta}{2}\right)e^{i\,(\mathrm{RA})}$$

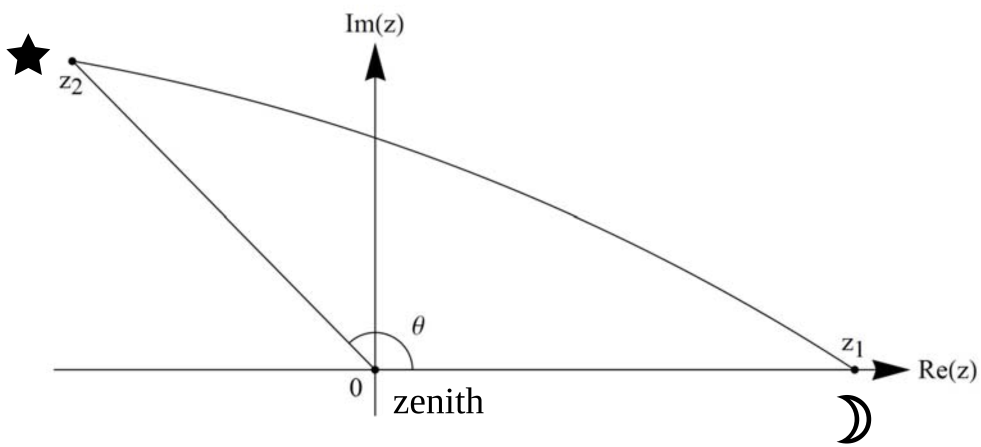
$$z_p = i\,\frac{2\,\mathrm{Im}\left(\bar{z}_1 z_2\right) \pm \left|\left(z_1 - z_2\right)\left(1+\bar{z}_1 z_2\right)\right|}{\left(1-\left|z_2\right|^2\right)\bar{z}_1 - \left(1-\left|z_1\right|^2\right)\bar{z}_2}$$

$$\alpha = \arg(z); \quad \delta = 2\tan^{-1}\left|z\right| - \frac{\pi}{2}$$

in reference $\alpha = 10^{\mathrm{h}}31^{\mathrm{m}}, \quad \delta = +1^\circ$

Clearing Lunar Distances

- An exercise in spherical trigonometry
- Does not depend on spherical coordinates
  - complex numbers loose some of their advantages
- Choose to map the zenith to zero on the complex plane
  - i.e. complex plane is tangent to the sphere at the zenith point



$$\tan^2\frac{d}{2} = \frac{z_1^2 + z_2^2 - 2z_1z_2\cos\theta}{1 + z_1^2z_2^2 + 2z_1z_2\cos\theta} \qquad \cos\theta = \frac{z_1^2 + z_2^2 - \left(1 + z_1^2z_2^2\right)\tan^2\frac{d}{2}}{2z_1z_2\left(1 + \tan^2\frac{d}{2}\right)}$$

Where *d* = measured angular distance between the Moon and Star

$$|z_1| \equiv z_1 = \tan\left(\frac{ZD_{☽}}{2}\right); \quad |z_2| \equiv z_2 = \tan\left(\frac{ZD_{★}}{2}\right)$$

Formulas entirely in terms of tangents of the half lengths of the sides

The star α Pegasi is observed on 31st December, 1884 from Absarat, Nubia, Nile Valley

Wilberforce Clarke, H., *Longitude by Lunar Distances*, W. H. Allen & Co., London, 1885; http://books.google.com/books?id=mtoMAAAAYAAJ.

	Clearing Lunar Distance Sight								
Inputs	Apparent lunar distance, *d*	103	°	26	'	24	"
	Apparent lunar altitude, h_M	35	°	37	'	28	"
	Apparent stellar altitude, h_S	40	°	17	'	24	"
	Geocentric lunar altitude, h'_M	36	°	26	'	01	"
	Geocentric stellar altitude, h'_S	40	°	16	'	15	"
Calculations	$\lvert z_1 \rvert$	0.513661					
	$\lvert z_2 \rvert$	0.463230					
	$tan^2 d/2$	1.605615					
	$cos\ \theta$	-0.982350					
	$\lvert z'_1 \rvert$	0.504768					
	$\lvert z'_2 \rvert$	0.463433					
	$tan^2 d'/2$	1.561277					
Results	Geocentric lunar distance, *d'*	102° 39' 30.6"					

$$|z_1| \equiv z_1 = \tan\left(\frac{\mathrm{ZD}_{☽}}{2}\right); \quad |z_2| \equiv z_2 = \tan\left(\frac{\mathrm{ZD}_{\star}}{2}\right)$$

$$\cos\theta = \frac{z_1^2 + z_2^2 - \left(1 + z_1^2 z_2^2\right)\tan^2\frac{d}{2}}{2z_1 z_2\left(1+\tan^2\frac{d}{2}\right)}$$

$$\tan^2\frac{d'}{2} = \frac{z_1'^2 + z_2'^2 - 2z_1'z_2'\cos\theta}{1 + z_1'^2 z_2'^2 + 2z_1'z_2'\cos\theta}$$

in reference $d' = 102°39'30''$

Extensions and Generalizations

The rotational form

$$\mathrm{T}(z) = \frac{az+b}{-\bar{b}z+\bar{a}}$$

has deep connections to other fields. Coefficients *a* and *b* are related to

- Cayley-Klein coefficients that appear in quantum mechanics of spin ½ particles
- Quaternion representation of rotations used in video game algorithms

Bilinear form accommodates relativistically correct Lorentz boost

$$\mathrm{T}(z) = \frac{az+b}{\bar{b}z+d}; \quad a,d \in \mathrm{R}$$

- Simplifies calculation of annual aberration (≈ 0.3´ effect)

$$\tan\frac{\theta'}{2} = \sqrt{\frac{c-v}{c+v}}\tan\frac{\theta}{2}$$

- Pure Lorentz boost is formally a rotation through an imaginary angle!

Effect of aberration of light on distance between star pairs

EX-MERIDIAN STAR PAIRS,
WITH DISTANCES FOR EVERY TEN DAYS.
(See *Introduction*, p. xiv.)

EX-MERIDIAN STAR PAIRS,
WITH DISTANCES FOR EVERY TEN DAYS.
(See *Introduction*, p. xiv.)

III. **θ Scorpii** and **α Ophiuchi**. R.A. $17^h 30^m$, Dec. 42° 56′ S., Mag. 2·0 . R.A. $17^h 30^m$, Dec. 12° 38′ N., Mag. 2·1

IV. **α Pavonis** and **γ Cygni**. R.A. $20^h 18^m$, Dec. 57° 2′ S., Mag. 2·0 . R.A. $20^h 19^m$, Dec. 39° 57′ N., Mag. 2·3

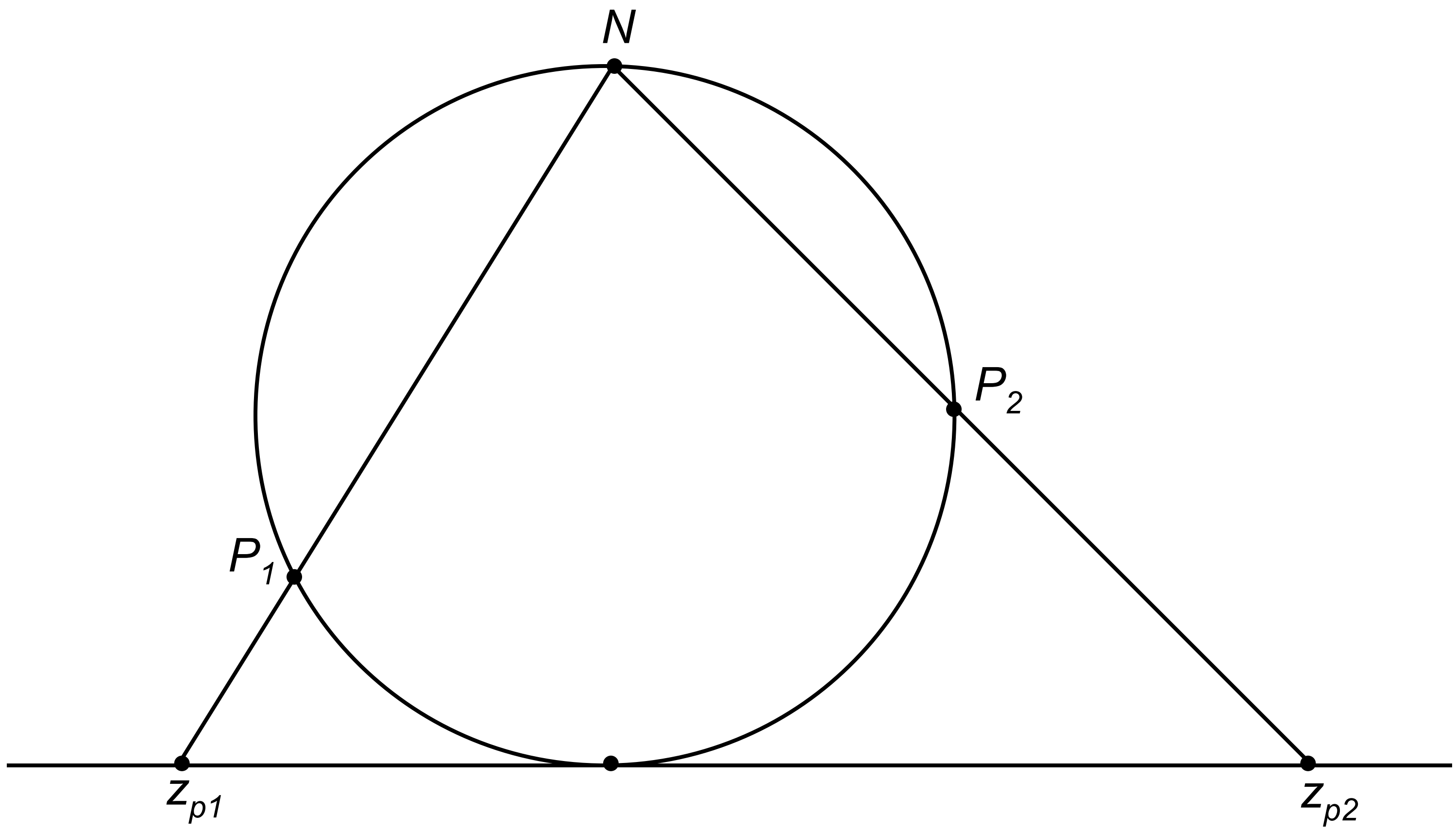
V. **Polaris** (α *Ursæ Minoris*) and **Alpheratz** (α *Andromedæ*). R.A. $1^h 24^m$, Dec. 88° 48′ N., Mag. 2·1 . R.A. $0^h 3^m$, Dec. 28° 34′ N., Mag. 2·1

VI. **Polaris** (α *Ursæ Minoris*) and **Schedir** (α *Cassiopeiæ*). R.A. $1^h 24^m$, Dec. 88° 48′ N., Mag. 2·1 . R.A. $0^h 35^m$, Dec. 56° 1′ N., Mag. 2·2—2·8

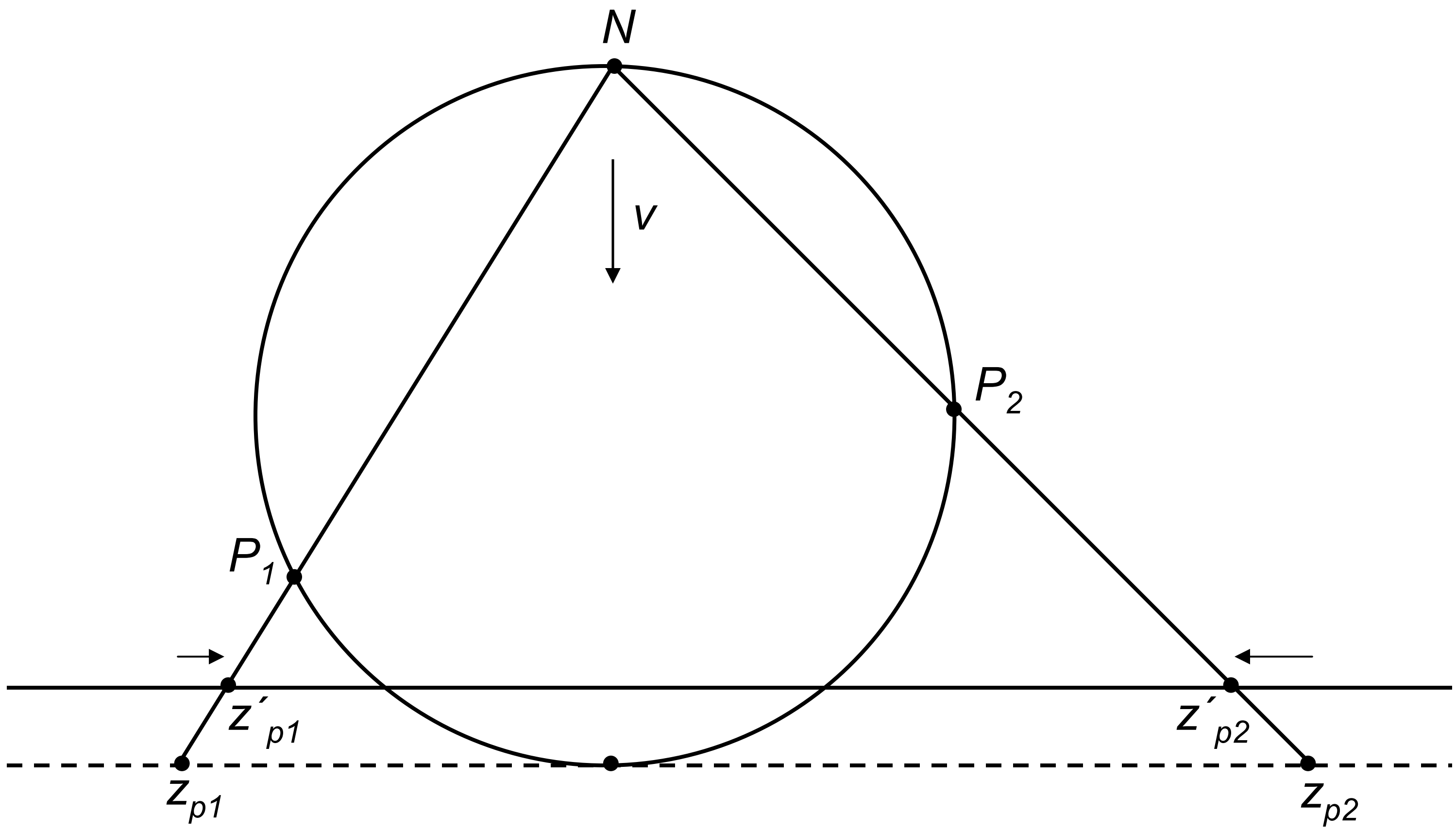
Date.		III. Distance.	IV. Distance.	V. Distance.	VI. Distance.
Jan.	1	55° 33′ 54″	96° 59′ 45″	60° 18′ 46″	32° 48′ 49″
	11	55 33 51	96 59 41	60 18 46	32 48 50
	21	55 33 48	96 59 35	60 18 46	32 48 50
	31	55 33 46	96 59 29	60 18 46	32 48 51
Feb.	10	55 33 43	96 59 24	60 18 47	32 48 51
	20	55 33 42	96 59 19	60 18 46	32 48 51
Mar.	1	55 33 41	96 59 14	60 18 45	32 48 51
	11	55 33 41	96 59 10	60 18 43	32 48 50
	21	55 33 41	96 59 7	60 18 41	32 48 50
	31	55 33 42	96 59 5	60 18 39	32 48 49
Apr.	10	55 33 43	96 59 3	60 18 37	32 48 49
	20	55 33 45	96 59 2	60 18 35	32 48 48
	30	55 33 47	96 59 2	60 18 33	32 48 47
May	10	55 33 49	96 59 3	60 18 31	32 48 46
	20	55 33 52	96 59 5	60 18 29	32 48 45
	30	55 33 55	96 59 7	60 18 27	32 48 44
June	9	55 33 58	96 59 10	60 18 25	32 48 43
	19	55 34 1	96 59 14	60 18 24	32 48 42
	29	55 34 4	96 59 18	60 18 23	32 48 41
July	9	55 34 7	96 59 23	60 18 22	32 48 40
	19	55 34 10	96 59 28	60 18 21	32 48 40
	29	55 34 12	96 59 33	60 18 21	32 48 40
Aug.	8	55 34 14	96 59 38	60 18 21	32 48 39
	18	55 34 16	96 59 43	60 18 22	32 48 39
	28	55 34 17	96 59 47	60 18 23	32 48 39
Sept.	7	55 34 18	96 59 51	60 18 24	32 48 40
	17	55 34 18	96 59 55	60 18 26	32 48 40
	27	55 34 18	96 59 58	60 18 28	32 48 41
Oct.	7	55 34 17	97 0 0	60 18 30	32 48 41
	17	55 34 16	97 0 1	60 18 33	32 48 42
	27	55 34 14	97 0 2	60 18 35	32 48 43
Nov.	6	55 34 11	97 0 2	60 18 37	32 48 44
	16	55 34 9	97 0 1	60 18 39	32 48 45
	26	55 34 6	96 59 59	60 18 41	32 48 46
Dec.	6	55 34 2	96 59 56	60 18 43	32 48 47
	16	55 33 59	96 59 52	60 18 45	32 48 48
	26	55 33 55	96 59 47	60 18 46	32 48 49
	36	55 33 52	96 59 43	60 18 46	32 48 50

Sprigge, J. A., Doak, W. F., Hudson, T. C. & Cox, A. S., *Stars and Sextants*, J. D. Potter, London, 1903 http://www.archive.org/details/starssextants00spri

Relativistic Aberration under Stereographic Projection



Relativistic Aberration under Stereographic Projection



Aberration moves star positions toward direction of observer's motion

- Equivalent to shifting the plane of stereographic projection
- Construction is exactly correct for special relativity
- Does not work for classical Bradley aberration

Summary and Conclusions

- Representing points in spherical coordinate systems as complex numbers provides an efficient and transparent way of performing calculations needed in Celestial Navigation
  - Involves only the basic arithmetic of complex numbers available on scientific calculators and computer languages
  - Circles on the sphere remain circles on the plane
  - Transforms many problems from trigonometric to algebraic
- Connected in fundamental ways with
  - Theory of conformal mappings
  - Rotations in 3D
  - Lorentz Transformations
- Advantages may be limited for problems not involving spherical coordinates
- Surprising that greater practical use has not been made of these methods

References

- Stuart, R. G., *Quarterly Journal of the Royal Astronomical Society* **25** (1984) 126 (http://articles.adsabs.harvard.edu/full/1984QJRAS..25..126S)
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- Stuart, R. G., *NAVIGATION: Journal of the Institute of Navigation*, **56** (2009) 221(preprint http://www.fer3.com/arc/m2.aspx?i=110015&y=200910)