Moon's Parallax

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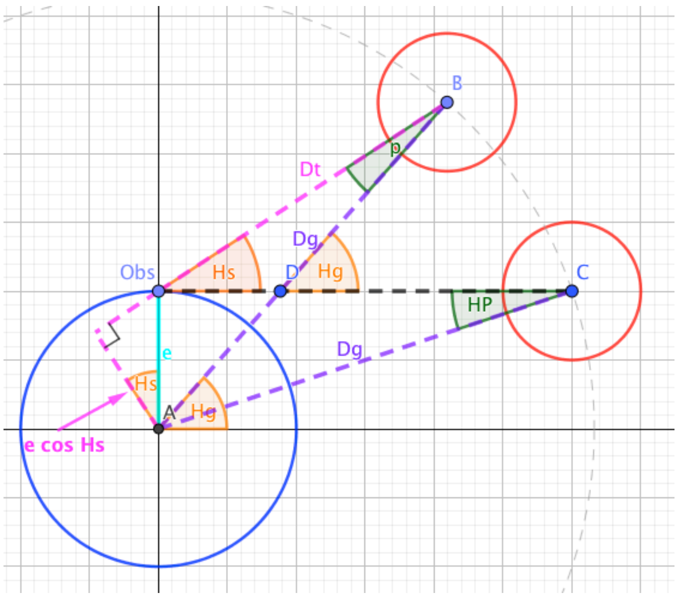
The altitude of the moon measured at the earth's center is different from the altitude measured by an observer at the earth's surface. In the figure, *Hg* (*g* for geocentric) is the angle formed by two lines: (1) a horizontal line parallel to the observer's horizon, but going through the center of the earth (point A), and (2) the line connecting the center of the earth (point A) and the center of the moon (point B). By corresponding angles (of a transversal crossing two parallel lines), *Hg* is also the angle at point D in the figure.



On the other hand, the observer measures angle *Hs* (*s* for sextant) between the horizon and the moon's center (in practice, we would measure to a limb and then add in the semidiameter of the moon). Note that *Hg* is an external angle of triangle formed by the points Observer, B and D, and thus equals the sum of the two non-adjacent angles:

$$Hg = Hs + p \qquad (1)$$

Angle *p* is the "parallax in altitude".

We next consider the small triangle formed by points Observer, A, and the unlabeled right angle. This right triangle has a hypotenuse equal to the earth's radius, labeled *e*. As shown, the angle between hypotenuse *e* and the adjacent side is *Hs*, which we deduce by noting *Hs* at the Observer plus a right angle (by construction, side *e* is perpendicular to side Observer-C), plus the unlabeled angle of the small triangle sum to a straight line equal to 180°:

$$\begin{aligned} Hs + 90^\circ + x &= 180^\circ \\ x &= 90^\circ - Hs \end{aligned} \qquad (2)$$

And because angle *x* and the center angle must add to 90° in the small right triangle, the center angle is *Hs*:

$$x + y = 90°$$
$$90° - Hs + y = 90° \tag{3}$$
$$y = Hs$$

In the small triangle, we have:

$$\cos Hs = \frac{adjacent}{hypotenuse} = \frac{s}{e} \tag{4}$$
$$s = e\cos Hs$$

where *e* is the size of the earth's radius, and *s* is the labeled side.

We now look at the bigger right triangle, which uses the same right angle as the small triangle, but with hypotenuse AB. By trigonometry:

$$\sin p = \frac{opposite}{hypotenuse} = \frac{e\cos Hs}{Dg} \tag{5}$$

where *Dg* is the distance between centers.

If the moon were exactly on the horizon (ignoring any refraction... this is only geometry), we form a third right triangle with points A, Observer and C. In this triangle we have:

$$\sin HP = \frac{opposite}{hypotenuse} = \frac{e}{Dg} \tag{6}$$

where *e* is the radius of the earth, *Dg* is the distance between bodies, and *HP* is the horizontal parallax, which is labeled in the figure.

Using (6) in (5):

$$\sin p = \frac{e\cos Hs}{Dg} = \sin HP\cos Hs \tag{7}$$

Equation (7) tells us how to find the parallax in altitude *p*, if we have a sextant altitude and (from ephemeris) the horizontal parallax.

Though the sine of an angle is defined as a ratio of two sides, Newton found an infinite series definition:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{8}$$

where angles are measured in radians. (Radians are dimensionless numbers.) For small values, x<<1 radian, the higher order terms are small, and we have the useful approximation:

$$\sin x \approx x \qquad (x \text{ in radians}) \tag{9}$$

For example,

$$\begin{aligned} 1^\circ &= \frac{\pi}{180} = 0.0174532925 \; rads \\ \sin 1^\circ &= 0.0174524064 \end{aligned} \tag{10}$$

For the small angles involved in parallax, generally smaller than one degree, we can use (9) in (7) to get a reasonably accurate expression:

$$p[rad] \approx HP[rad] \cos Hs \tag{11}$$

where the angles are expressed in radians. <u>However</u>, if we multiply both sides by a factor to convert from radians to, for example, arc minutes:

$$\begin{aligned} \left(\frac{360 \cdot 60}{\pi}\right) \cdot p[rad] &\approx \left(\frac{360 \cdot 60}{\pi}\right) \cdot HP[rad] \cos Hs \\ p['] &= HP['] \cos Hs \end{aligned} \tag{12}$$

the equation still holds.

Change Rates

The change in *Hg* with time is related to the change in *Hs*. Beginning with (1):

$$Hg = Hs + p \tag{13}$$

and taking the derivative with respect to time:

$$\frac{dHg}{dt} = \frac{dHs}{dt} + \frac{dp}{dt} \tag{14}$$

For the last term in (14), we use (7):

$$\frac{d \sin p}{dt} = \frac{d}{dt}(\sin HP \cos Hs) \tag{15}$$

That becomes

$$\cos p \frac{dp}{dt} = \cos HP \cos Hs \frac{dHP}{dt} - \sin HP \sin Hs \frac{dHs}{dt} \tag{16}$$

or,

$$\frac{dp}{dt} = \frac{\cos HP \cos Hs}{\cos p} \frac{dHP}{dt} - \frac{\sin HP \sin Hs}{\cos p} \frac{dHs}{dt} \tag{17}$$

That is the derivative in all of its glory, but if we assume the *HP* is constant or very slowly varying the first member can be dropped. If we assume both the horizontal parallax and the parallax in altitude are very small (in radians), we can use approximations (note cos $x \approx 1\text{-}x^2/2$, or in this case, just 1):

$$\frac{dp}{dt} \approx -HP \sin Hs \frac{dHs}{dt} \tag{18}$$

If we want the units to be, for example, arc minutes per second, we multiply both sides by a conversion factor. However, in (18), this factor can only be applied once to both sides, and the *HP* (which is an approximation for sin *HP*) must remain in radians. Putting (18) into (14), we have

$$\frac{dHg}{dt} = \frac{dHs}{dt} - HP \sin Hs \frac{dHs}{dt}$$
$$\frac{dHg}{dt} = (1 - HP \sin Hs) \frac{dHs}{dt} \tag{19}$$

Here, the change in the geocentric altitude, and the change in sextant altitude can both be in units of arc minutes per second, but *HP* must be in radians (or sin *HP* used instead of simply *HP*).

If one or the other of the rates is given, (19) can be used to deduce the rate of the other.