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Applications of Complex Analysis to Spherical Coordinate Geometry

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SUMMARY

Transformations between systems of spherical coordinates are obtained by mapping the sphere by stereographic projection on to the complex number plane. Rotations of the sphere can be identified with a particular class of bilinear transformations on the complex plane. A number of results pertaining to the images of great circles under stereographic projection is also derived.

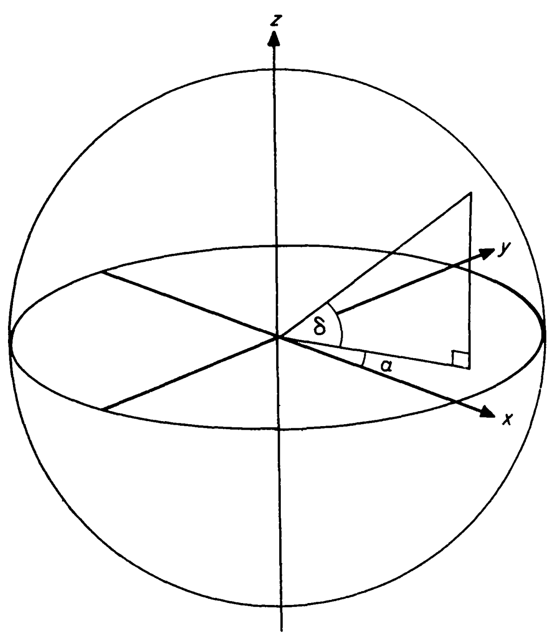
1 INTRODUCTION

Rotations and other manipulations of 2-dimensional spherical coordinate systems of the type shown in Fig. 1 are usually performed by first converting to Cartesian coordinates via the equations

$$x = \cos\delta \cos\alpha$$
$$y = \cos\delta \sin\alpha$$
$$z = \sin\delta.$$

Having done this the full power of 3-dimensional vector analysis is at our disposal. Rotations are obtained by matrix multiplications (3), and other useful operations, such as the vector cross-product, are easily performed.

However this approach was thought to be unsatisfactory in that it introduces a third coordinate to what is essentially a 2-D problem. As a consequence the number of equations required for any given operation is



increased which may be a serious disadvantage if the calculations are performed on a machine with limited storage such as a programmable calculator.

This paper describes a method by which the usual operations of vector analysis can be performed in a purely 2-D framework. The spherical coordinate system is identified with the Riemann sphere and is mapped by stereographic projection on to the complex plane. The resulting complex numbers when expressed in polar form clearly display the original spherical coordinates. Almost all operations can now be performed without further reference to the sphere and without the introduction of an unnecessary third coordinate. In addition, some interesting properties of the images of great circles under stereographic projection will be shown to arise as a consequence of this approach.

2 STEREOGRAPHIC PROJECTION

Fig. 2 shows a sphere of unit diameter (the Riemann sphere) tangent to the complex plane at the origin O; N is the point on the sphere diametrically opposite O. With a given point A on the sphere we associate the complex number A′ which lies at intersection of the ray $\overline{NA}$ and complex plane. In this way the surface of the sphere is mapped one-to-one on to the complex plane. The mapping can be shown to be conformal and is called stereographic projection.

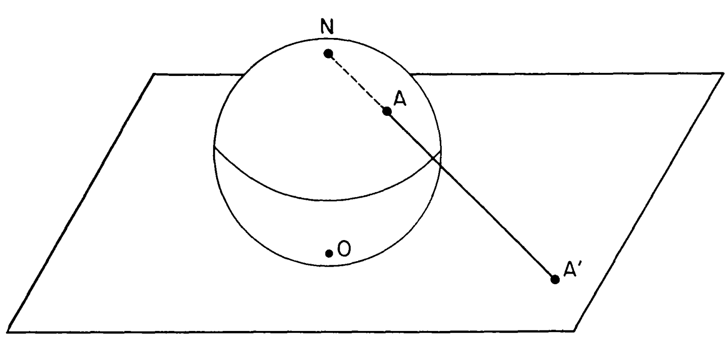
Suppose now that the sphere has a spherical coordinate system of the type shown in Fig. 1 imposed upon it in such a way that the 'prime meridian' (the meridian of zero longitude) is mapped on to the positive real axis and the 'south pole' is coincident with O. From simple geometry it follows that the point on the sphere with coordinates α, δ maps on to the complex number, $re^{i\phi}$ where

$$r = \tan\left(\frac{\delta}{2} + \frac{\pi}{4}\right) \tag{2.1}$$

$$\phi = \alpha. \tag{2.2}$$

Evidently the image of the 'equator' on the sphere is the unit circle on the complex plane.

All the terminology associated with spherical coordinate systems can now be carried over to the complex plane. For example a great circle on the complex plane (or simply great circle where there is no ambiguity) is taken to mean a curve that is the image of a great circle on the sphere



under stereographic projection. Similarly the angular distance between two complex numbers, z_1 and z_2 is interpreted as the angle subtended at the centre of the sphere by the two points which have z_1 and z_2 as their images under stereographic projection.

In the spirit of the above we consider:

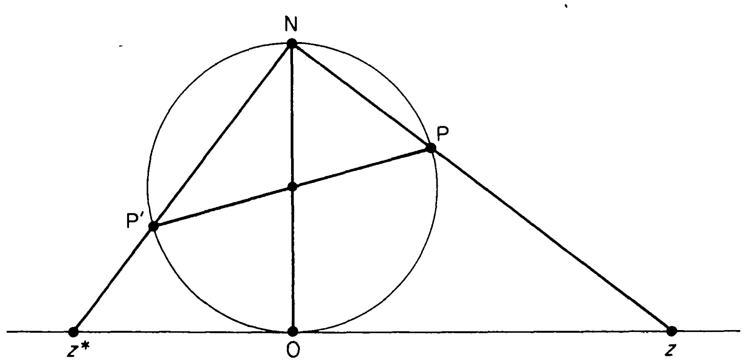
2.1 *Diametrically opposite points on the complex plane.* Let z be any point on the complex plane and let z^* be diametrically opposite. According to Fig. 3

$$\arg z^* = \arg z + \pi$$

$$|z z^*| = 1$$

and hence

$$z^* = -\frac{1}{\bar{z}}.$$



3 BILINEAR ROTATIONS ON THE COMPLEX PLANE

The term bilinear transformation is used to designate a rational function $w = w(z)$ of first order. Its general form is

$$w = T(z) = \frac{a z + b}{c z + d}$$

where a, b, c and d are complex constants and $ad-bc \neq 0$.

It turns out (1) that any rotation of the Riemann sphere about a diameter can be represented by a bilinear transformation on the complex plane. The crucial point here is that under such a rotation pairs of diametrically opposite points are mapped on to pairs of diametrically opposite points. Thus if

$$w = \frac{a z + b}{c z + d} \tag{3.1}$$

then from the last section we must also have

$$-\frac{1}{\bar{w}} = \frac{-(a/\bar{z}) + b}{-(c/\bar{z}) + d}$$

which gives

$$w = \frac{\bar{d} z - \bar{c}}{-\bar{b} z + \bar{a}}. \tag{3.2}$$

Using (3.1) and (3.2) it follows that

$$\frac{a}{\bar{d}} = \frac{-b}{\bar{c}} = \frac{-c}{\bar{b}} = \frac{d}{\bar{a}}\,.$$

Evidently

$$|\,a\,| = |\,d\,|$$

and therefore

$$d = \lambda\,\bar{a} \qquad \text{where} \quad |\,\lambda\,| = 1.$$
$$c = -\lambda\bar{b}$$

Since we can write $1/\lambda$ as the quotient of two conjugate complex numbers, $1/\lambda = \mu/\bar{\mu}$ we finally obtain

$$w = T(z) = \frac{a\,z + b}{-\bar{b}\,z + \bar{a}}. \tag{3.3}$$

The converse, that any bilinear transformation of the form of (3.3) represents a rotation of the Riemann sphere, is also true but the proof is outside the scope of this paper. The interested reader is referred to Nevanlinna & Paatero (1) or a similar text on complex analysis.

Bilinear transformations of the form of (3.3) will henceforth be referred to as bilinear rotations.

3.1 *Angular distance between points on the complex plane.* Let z_1 and z_2 be two points on the complex plane. To find their angular separation, θ say, we make use of the bilinear rotation

$$T(z) = \frac{z - z_1}{\bar{z}_1 z + 1}$$

for which

$$T(z_1) = 0$$

$$T(z_2) = \frac{z_2 - z_1}{\bar{z}_1 z_2 + 1}.$$

As the angular distance between the points is invariant under rotations it follows from (2.1) that

$$\theta = 2\tan^{-1}\left|\frac{z_2 - z_1}{\bar{z}_1 z_2 + 1}\right|. \tag{3.4}$$

4 GREAT CIRCLES ON THE COMPLEX PLANE

4.1 *Equation of a great circle on the complex plane.* In Section 2 we noted that the equator of the sphere is mapped on to the unit circle by stereographic projection. Since the equator can be mapped on to any great circle on the sphere by a suitable rotation it follows that the unit circle can be mapped on to any great circle on the complex plane by an equivalent bilinear rotation. Bilinear transformations take circles into circles (1) and hence great circles on the complex plane are indeed circles. (Note that here a straight line is regarded as a circle of infinite radius.)

Any circle on the complex plane has an equation of the form

$$| z - z_0 |^2 = \varrho^2$$

where z_0 is its centre and ϱ is its radius. This can be expanded to

$$z\bar{z} - \bar{z}_0 z - z_0\bar{z} + | z_0 |^2 - \varrho^2 = 0. \qquad (4.1)$$

For any point z on a given great circle we require that the point $-(1/\bar{z})$ also lies on the same great circle since z and $-(1/z)$ are diametrically opposite. Therefore

$$\frac{1}{z\,\bar{z}} + \frac{\bar{z}_0}{\bar{z}} + \frac{z_0}{z} + | z_0 |^2 - \varrho^2 = 0$$

or

$$-1 - \bar{z}_0 z - z_0\bar{z} + (\varrho^2 - | z_0 |^2)\, z\bar{z} = 0. \qquad (4.2)$$

Comparison of the coefficients in (4.1) and (4.2) yields the relation

$$\varrho^2 = 1 + | z_0 |^2. \qquad (4.3)$$

This shows that any great circle is uniquely determined by its centre z_0. We should stress here that the centre of a great circle is a concept applicable only to the complex plane and has no meaningful equivalent on the sphere.

4.2 *Results concerning the centres and poles of great circles*

4.2.1 Centre of a great circle defined by two points. Suppose that z_1 and z_2 are two distinct complex numbers but that $z_1 \neq -(1/\bar{z}_2)$, i.e. z_1 is not diametrically opposite z_2. The centre z_0 of the great circle on which z_1 and z_2 lie satisfies the equations

$$z_1\bar{z}_1 - \bar{z}_0 z_1 - z_0\bar{z}_1 - 1 = 0$$
$$z_2\bar{z}_2 - \bar{z}_0 z_2 - z_0\bar{z}_2 - 1 = 0.$$

By eliminating $\bar{z}_0$ one obtains

$$z_0 = \frac{(|z_1|^2 - 1)z_2 - (|z_2|^2 - 1)z_1}{\bar{z}_1 z_2 - z_1\bar{z}_2}.$$

4.2.2 Relationship between the centre and poles of a great circle on the complex plane. Suppose that for a given point z_1 we require to find the great circle which has z_1 for a pole. In other words z_1 lies at right angles to all points on the required circle. From (3.4) we see that the points z on the circle must satisfy

$$\left| \frac{z - z_1}{\bar{z}_1 z + 1} \right| = 1$$

which after a little algebra gives

$$z\,\bar{z} - \frac{2}{1 - | z_1 |^2}(z_1\bar{z} + \bar{z}_1 z) - 1 = 0.$$

This is the equation of a circle with centre

$$z_0 = \frac{2z_1}{1 - | z_1 |^2}. \qquad (4.4)$$

Conversely, if z_0 is the centre of a great circle, (4.4) may be inverted to obtain its poles. From the above we have

$$z_0 z_1 \bar{z}_1 + 2z_1 - z_0 = 0.$$

Taking the conjugate of this equation and eliminating $\bar{z}_1$ yields

$$\bar{z}_0 z_1^2 + 2z_1 - z_0 = 0.$$

This quadratic has solutions

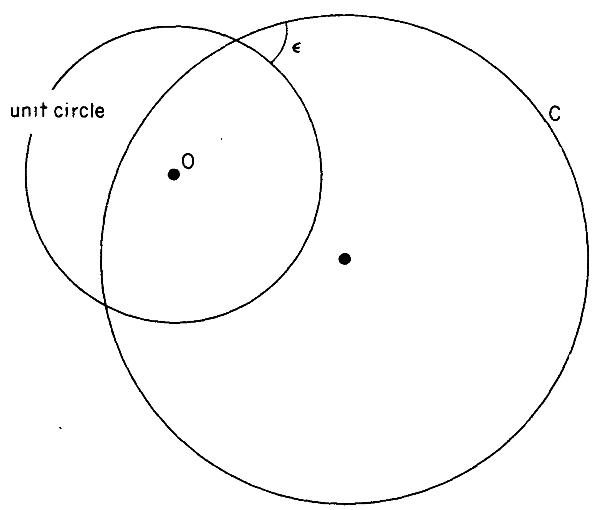
$$z_1 = \frac{-1 \pm \sqrt{1 + |z_0|^2}}{\bar{z}_0} = \frac{-1 \pm \varrho}{\bar{z}_0}$$

which reflects the fact that every great circle has two poles.

4.2.3 Relationship between the inclination and radius of a great circle. Suppose that C is a great circle of radius r and that ε is the angle of intersection of C with the unit circle (Fig. 4). The cosine rule of plane trigonometry and (4.3) may be used to obtain

$$\cos \varepsilon = 1/r.$$

Since stereographic projection is conformal, this result means that any great circle on the complex plane of radius $r = \sec \varepsilon$ is the image of a great circle on the sphere with inclination ε to the equator.



5 COEFFICIENTS FOR BILINEAR ROTATIONS

We now derive expressions for the complex coefficients a and b of bilinear rotations in terms of parameters which are normally used to specify a rotation.

5.1 *Rotation by an angle θ about a diameter.* As in Fig. 3, P and P' are a pair of diametrically opposite points on the sphere whose images under stereographic projection are the complex numbers z_1 and $-(1/\bar{z}_1)$ respectively. The transformation we obtain will have the effect of rotating the coordinate axes by an angle θ in the positive right-handed sense about the vector $\overrightarrow{P'P}$.

The bilinear rotation

$$U(z) = \frac{z - z_1}{\bar{z}_1 z + 1} \tag{5.1}$$

maps z_1 on to 0 and $-(1/\bar{z}_1)$ on to ∞.

If $w = T(z)$ is the required transformation then

$$U(w) = e^{i\theta}U(z)$$

since multiplication by $e^{i\theta}$ rotates all points by an angle θ in the correct sense about the origin.
Hence

$$w = T(z) = U^{-1}e^{i\theta}U(z).$$

Substituting (5.1) into the above gives

$$a = -e^{\frac{i\theta}{2}} - |z_1|^2 e^{\frac{-i\theta}{2}}$$

$$b = z_1(e^{\frac{i\theta}{2}} - e^{\frac{-i\theta}{2}})$$

$$= 2iz_1 \sin\frac{\theta}{2}.$$

5.2 *Rotations determined by two points and their images.* Suppose that we wish to find the coefficients of the bilinear rotation $w = T(z)$ for which

$$w_1 = T(z_1)$$
$$w_2 = T(z_2)$$

where

$$\left|\frac{z_2 - z_1}{\bar{z}_1 z_2 + 1}\right| = \left|\frac{w_2 - w_1}{\bar{w}_1 w_2 + 1}\right|.$$

We define the following bilinear rotations:

$$U(z) = \frac{z - z_1}{\bar{z}_1 z + 1}$$

which maps z_1 on to 0

and

$$V(w) = \frac{w - w_1}{\bar{w}_1 w + 1}$$

which maps w_1 on to 0.

If λ is the constant of unit modulus which satisfies the condition

$$V(w_2) = \lambda U(z_2)$$

we see that the required rotation is

$$w = T(z) = V^{-1}\lambda U(z)$$

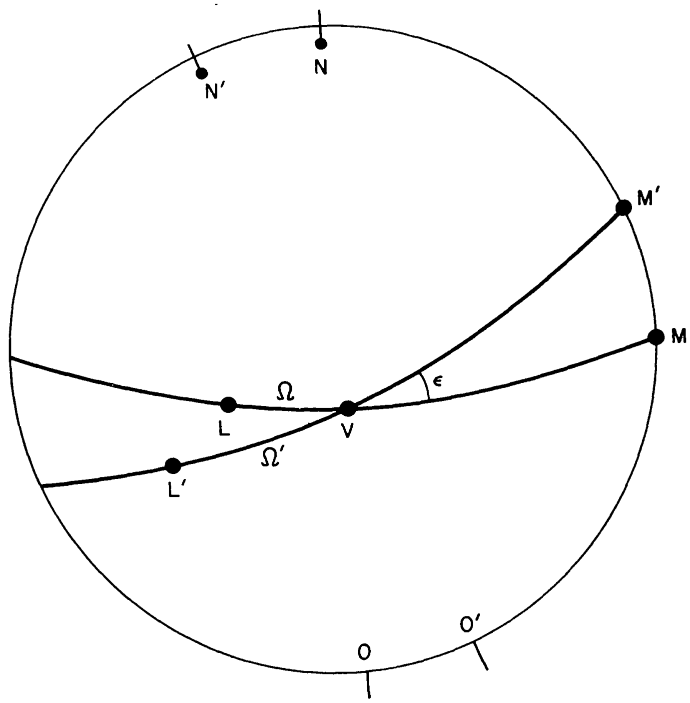
where

$$\lambda = V(w_2)/U(z_2)$$

$$= \frac{(w_2 - w_1)(\bar{z}_1 z_2 + 1)}{(\bar{w}_1 w_2 + 1)(z_2 - z_1)}.$$

Therefore

$$a = \lambda^{1/2} + w_1\bar{z}_1\lambda^{-1/2}$$
$$b = w_1\lambda^{-1/2} - z_1\lambda^{1/2}.$$



5·3 *Rotation determined by the ascending node and inclination of the equator.* In Fig. 5 L lies on the prime meridian and LM on the equator of our original coordinate system. Similarly L′ lies on the prime meridian and L′M′ on the equator of our new coordinate system. Here V is the point of intersection of L′M′ with LM and is thus the ascending node of the new equator with respect to the first. As indicated ε is the angle of intersection of LM and L′M′ at V and Ω, Ω′ are the angles subtended at the centre of the sphere by the pairs of points L, V and L′, V respectively.

From the figure we see that the original coordinates of the point o are

$$\alpha_1 = 0;\ \delta_1 = \frac{-\pi}{2}$$

and that in the new system they are

$$\alpha_1' = \frac{-\pi}{2} + \Omega';\ \delta_1' = \frac{-\pi}{2} + \varepsilon.$$

The original coordinates of the point o′ are

$$\alpha_2 = \frac{\pi}{2} + \Omega;\ \delta_2 = \frac{-\pi}{2} + \varepsilon.$$

and in the new system they are

$$\alpha_2' = 0;\ \delta_2' = \frac{-\pi}{2}.$$

By finding the images of these points under stereographic projection we see that we require a bilinear rotation which maps

$$\text{o on to } \tan\frac{\varepsilon}{2}\, e^{i(\Omega' - \frac{\pi}{2})}$$

$$\tan\frac{\varepsilon}{2}\, e^{i(\Omega + \frac{\pi}{2})} \text{on to o.}$$

Using the results of the previous section we easily obtain

$$a = e^{i\left(\frac{\Omega' - \Omega}{2}\right)}$$

$$b = \tan\frac{\varepsilon}{2}\, e^{i\left(\frac{-\pi}{2} + \frac{\Omega + \Omega'}{2}\right)}.$$

5.4 *Example* (**2**). For convenience we will work in degrees for this example. The coordinates of a point A are $\alpha = 75°$, $\delta = 15°$. Find the coordinates of A in a system related to the original by the parameters

$$\varepsilon = 23{\cdot}5°$$
$$\Omega = 215°$$
$$\Omega' = 115°.$$

Solution. From the last section the coefficients of the required bilinear rotation are

$$a = e^{i(-50°)} = 0{\cdot}6428 - 0{\cdot}7660i$$
$$b = 0{\cdot}2080e^{i(75°)} = 0{\cdot}0538 + 0{\cdot}2009i.$$

Under stereographic projection the point $\alpha = 75°$, $\delta = 15°$ is mapped to

$$z = 1{\cdot}3032e^{i(75°)} = 0{\cdot}3373 + 1{\cdot}2588i.$$

Substituting these values into (3.3) we obtain

$$T(z) = 1{\cdot}4274 - 0{\cdot}9195i = 1{\cdot}6979e^{i(-32{\cdot}79°)}.$$

By taking the inverse of (2.1) and (2.2) we find that in the new coordinate system

$$\alpha' = 327{\cdot}21°,\ \delta' = 29{\cdot}01°.$$

6 SPHERICAL TRIGONOMETRY

We now demonstrate that fundamental formulae of spherical trigonometry can be derived from plane trigonometry and previous results in this paper.

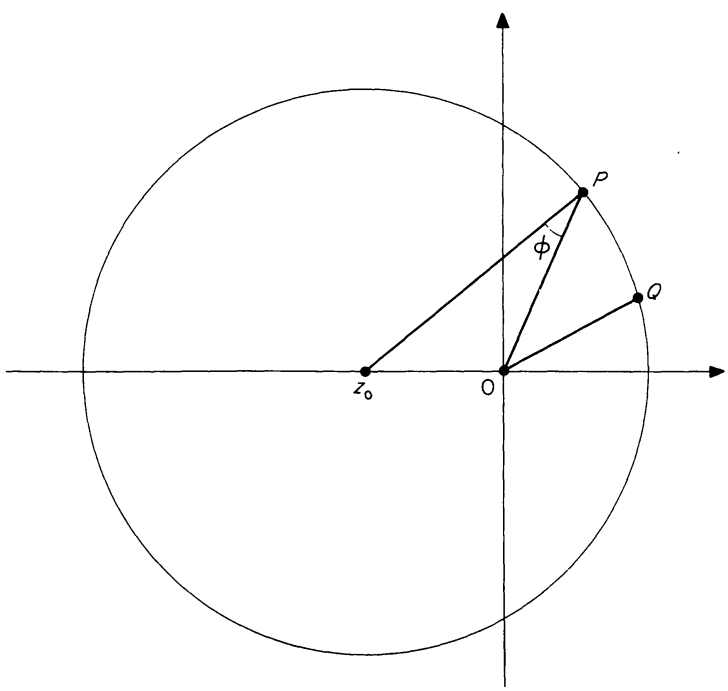


Fig. 6 shows a great circle on the complex plane with centre z_0 and radius ϱ where $\varrho^2 = |z_0|^2 + 1$. P and Q are distinct complex numbers lying on the great circle. We note that the curvilinear triangle OPQ is the image of a spherical triangle under stereographic projection as each of its sides is the arc of a great circle on the complex plane.

The cosine rule of plane trigonometry gives

$$\cos \phi = \frac{\varrho^2 - |z_0|^2 + |P|^2}{2|P|\varrho}$$

or

$$\sin P = \frac{1 + |P|^2}{2|P|\varrho}$$

where sin P conventionally denotes the sine of interior angle between sides of the triangle which meet at P.
Similarly

$$\sin Q = \frac{1 + |Q|^2}{2|Q|\varrho}$$

and hence

$$\frac{\sin Q}{\left(\dfrac{2|P|}{1 + |P|^2}\right)} = \frac{\sin P}{\left(\dfrac{2|Q|}{1 + |Q|^2}\right)}. \tag{6.1}$$

If the angular distance between the origin O and P is θ_p then (3.4) gives

$$|P| = \tan \frac{\theta_P}{2}$$

which yields

$$\sin \theta_p = \frac{2|P|}{1 + |P|^2}. \tag{6.2}$$

Note that θ_p is the angle subtended at the centre of the sphere by the corresponding side of the spherical triangle.

Similarly if θ_Q is the angular distance between the origin and Q it follows that

$$\sin \theta_Q = \frac{2|Q|}{1 + |Q|^2}. \tag{6.3}$$

Hence combining (6.1), (6.2) and (6.3) we have

$$\frac{\sin P}{\sin \theta_q} = \frac{\sin Q}{\sin \theta_p}.$$

The conformality of stereographic projection now guarantees that for any spherical triangle ABC,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

where we have used the usual conventions to denote the sides and angles of a spherical triangle.

Evidently the other fundamental formulae of spherical trigonometry can be derived in a similar fashion.

7 CONCLUSION

In the foregoing paper we have demonstrated the usefulness of stereographic projection in treating problems encountered in the use of spherical coordinate systems.

Other problems of practical importance can also be dealt with by a similar approach. For example the effect on the observed coordinates of stars due to the aberration of light can be treated by a combination of bilinear rotations and non-analytic transformations of the complex plane.

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