Applications of Complex Analysis to Celestial Navigation

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Abstract

Mapping points on the sphere by stereographic projection to points on the plane of complex numbers transforms the spherical trigonometry calculations, performed in course of celestial navigation, into arithmetic operations on complex numbers. Rotations on spherical coordinate systems take a simple bilinear form. Relationships between circles on the sphere and circles on the complex plane are derived. The results are applied to the reduction of double altitude sights, altitude and azimuth determination and the clearing of lunar distance sights.

Introduction

Celestial navigation involves the measurement of altitudes of the Sun, Moon, planets or stars, above some reference horizon or, as in the case of longitude by lunar distances, the separation between a pair of such bodies. Measurements must be corrected for a number of effects, horizon dip, refraction, parallax etc., before being used to compute the observer's true position. These sight reduction calculations generally employ formulas based on classical spherical trigonometry [1, 2]. Traditionally, such formulas were often arranged in a way that minimized the labor required to evaluate them by means of logarithm tables. Today this is less of a concern with simplicity, compactness and transparency assuming greater importance.

In this paper it is shown that formulas needed to perform standard sight reductions can be expressed without the explicit appearance of trigonometric functions by exploiting the intimate connection between trigonometry and complex numbers. This is achieved by stereographic projection of points on the terrestrial globe onto the plane of complex numbers. Significant simplification arises from the fact that the two angular coordinates, needed specify points on the sphere, are subsumed into a single complex number and do not have to be carried separately through calculations. The upshot is that formulas obtained by the approach described here are compact and transparent in nature. Many scientific calculators and computer languages, Fortran, C++, Perl etc., incorporate complex number operations as standard features and hence evaluation is straightforward in practice. Moreover spherical geometry is replaced by the more broadly intuitive plane geometry. Results will be derived and numerical examples given for reduction of double altitude sights, altitude and azimuth determination, as required in Line of Position (LOP) navigation, and the clearing of lunar distance sights.

In complex number theory, stereographic projection is used to establish an isomorphism between points on the complex plane and points lying on a sphere – the Riemann Sphere. Under stereographic projection, circles on the sphere map to circles or straight lines (circles of infinite radius) on the plane. It was shown in ref.[3] that if the Riemann Sphere is identified with the celestial sphere, then transformations between celestial spherical coordinates systems, e.g. equatorial to ecliptic, could be realized through arithmetic operations on complex numbers. Results concerning the properties of the images of great circles on the complex plane were also given. The results from ref.[3] that are needed here are summarized in the Appendix. In the present work the methods are extended to derive results that are applicable to problems in celestial navigation. In this case the terrestrial globe is identified Riemann sphere and stereographic projection maps points on the globe to points on the complex plane. While demonstrated specifically for celestial navigation, the results concerning circles on the sphere and underlying methodology in general may be applied to any problem in which the use of spherical coordinate geometry is required.

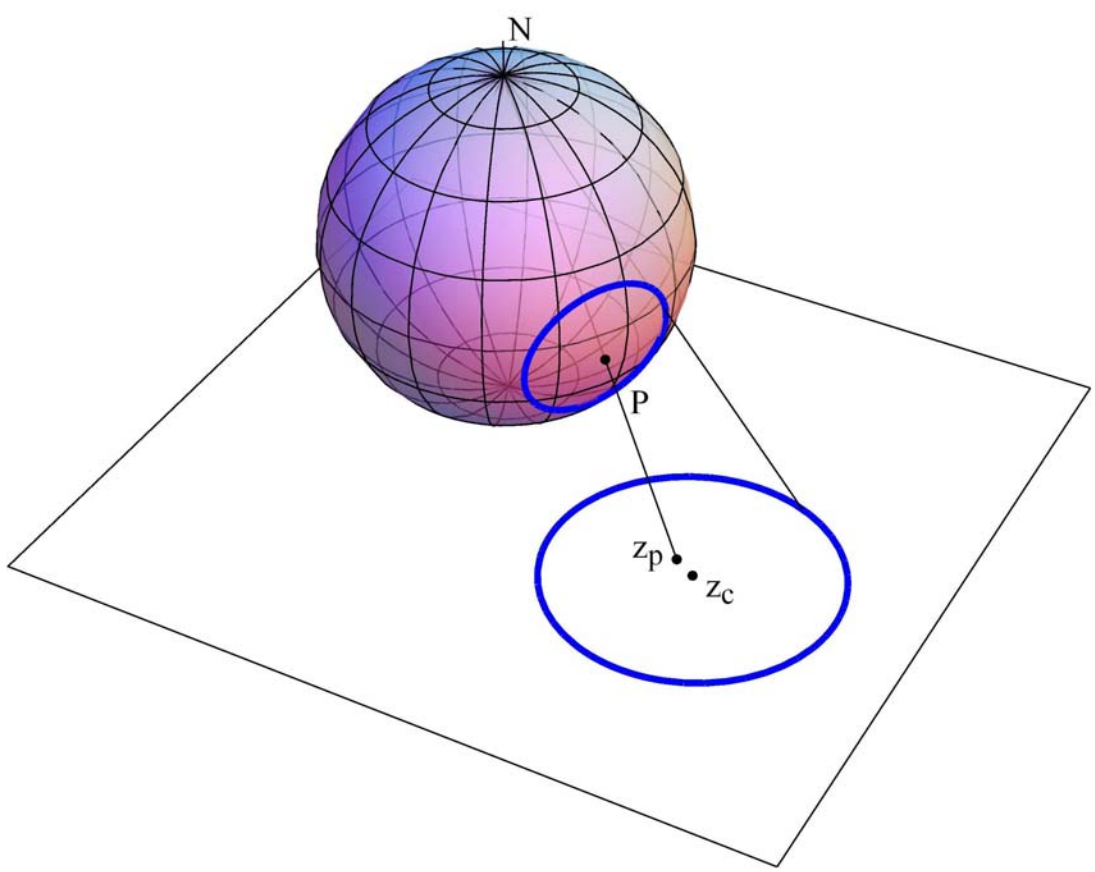

Figure 1. Stereographic projection of the terrestrial globe onto the complex number plane tangent to the sphere at its South pole. A circle on the sphere with *pole* at point P maps to circle on the sphere with center z_c . The image of the point P on the complex plane is z_p and is not, in general, coincident with z_c .

Stereographic projection has been applied elsewhere [4] to relate the identities of plane trigonometry to those of spherical trigonometry. In the field of astronautics [5], stereographic projection of the sphere onto the complex plane was used to derive the equations of motion of a rotating rigid body in terms of one complex and one real coordinate, (w, z).

The operation of stereographic projection is depicted in Fig.1. A sphere of unit diameter is tangent to the complex plane at its South Pole. A straight line is drawn from the North Pole, N, through a point, P, on the surface of the sphere and extended to its intersection with the complex plane. The complex number lying at the intersection, z_p, is the image of P under stereographic projection. If the coordinates of P are latitude, L, and longitude, λ, then

$$z_p = \tan\left(\frac{\pi}{4} + \frac{L}{2}\right) e^{i\lambda} \tag{1.1}$$

where the function arguments are expressed in radian measure. Under this projection the meridian of zero longitude maps to the positive real axis, and the South Pole of the sphere maps to zero on the complex plane. Longitude increases in an anticlockwise direction about the origin which matches the convention used in defining the arguments of complex numbers. Because of the singularity at $L = 90^\circ$, when considering positions very close to the North Pole, it may be convenient to consider an alternative form of the stereographic projection in which the roles of the North and South Poles in Fig.1 are reversed. In that case the projection

$$z_p = \tan\left(\frac{\pi}{4} - \frac{L}{2}\right) e^{-i\lambda} \tag{1.2}$$

may be used where the minus sign in the argument of the exponential function has been introduced to preserve the property that both longitude and complex number arguments increase in the same sense about the origin.

For navigation on or near the Earth's surface the two forms (1.1) and (1.2) suffice for most practical purposes. In other applications it may be necessary to arrange for the singularity to be located points at other than the poles [6].

Fig.1 also shows a circle on the sphere centered on P. The point P is referred to as a pole of this circle. Under stereographic projection the circle maps to a circle on the complex plane with center z_c. Note that $z_c \neq z_p$ except when P lies at the North or South Poles of the sphere.

Properties of Circles on the Complex Plane

The equation

$$|z - z_c|^2 = r^2 \tag{2.1}$$

describes a circle on the complex plane with center at z_c and radius r. It can be written

$$z\overline{z} - \overline{z}_c z - z_c \overline{z} + z_c \overline{z}_c - r^2 = 0\,. \tag{2.2}$$

The bar over a complex number here denotes its complex conjugate. From eq.(A.4) it follows that the circle on the sphere with pole P and angular radius θ is described by the equation

$$\tan^2\frac{\theta}{2} \equiv \rho^2 = \left|\frac{z - z_p}{1 + \bar{z}_p z}\right|^2 \tag{2.3}$$

that can be written

$$\left(1 - \rho^2 \left|z_p\right|^2\right) z\,\bar{z} - \left(1 + \rho^2\right)\left(z\,\bar{z}_p + \bar{z}\,z_p\right) + \left|z_p\right|^2 - \rho^2 = 0\,. \tag{2.4}$$

Upon some rearrangement and comparison with eq.(2.2), this can be seen to be a circle with center and radius

$$z_c = \frac{1 + \rho^2}{1 - \rho^2 \left|z_p\right|^2} z_p, \qquad r = \frac{1 + \left|z_p\right|^2}{\left|1 - \rho^2 \left|z_p\right|^2\right|} \rho\,. \tag{2.5}$$

For any great circle $\rho = 1$ and when its pole lies on the equator, $\left|z_p\right| = 1$, the denominators in eq.s(2.5) vanish yielding a circle of infinite radius. Such a circle is a meridian circle on the sphere and is represented by a straight line passing through the origin on the complex plane.

Conversely a circle on the complex plane with center z_c and radius r is the stereographic projection of a circle on the sphere with pole and angular radius

$$\begin{aligned} z_p &= \frac{z_c}{\left|z_c\right|}\left\{\frac{\left|z_c\right|^2 - r^2 - 1}{2\left|z_c\right|} \pm \sqrt{1 + \left(\frac{\left|z_c\right|^2 - r^2 - 1}{2\left|z_c\right|}\right)^2}\right\}, \\ \rho &= \pm\frac{r^2 - \left|z_c\right|^2 - 1}{2r} + \sqrt{1 + \left(\frac{r^2 - \left|z_c\right|^2 - 1}{2r}\right)^2} \end{aligned} \tag{2.6}$$

The two solutions in the above equations reflect the fact that a given circle on the sphere has two poles, P and P', lying diametrically opposite one another, and correspondingly two radii, ρ and ρ'. From the relationship between antipodal points, noted in the Appendix, it may be shown that, the poles and their radii satisfy the relations $\bar{z}_p z'_p = -1$ and $\rho\,\rho' = 1$. These conditions are evidently satisfied by eq.s(2.6). The eq.s(2.6) may be further verified by substituting the upper and lower sign solutions into eq.(2.5).

The properties of great circles presented in ref.[3] are recovered by setting $\rho = 1$.

Intersection points of two circles on the complex plane

Expressions for the Cartesian coordinates of the intersection points of two circles on a plane are well known and will not be rederived here. Instead the result is recast in a form that facilitates its evaluation directly using arithmetic operations on complex numbers.

Consider two circles on the complex plane with centers z_{c1}, z_{c2} and radii r_1, r_2 respectively. Writing $d = |z_{c1} - z_{c2}|$, the intersection points of the circles are

$$z = \frac{1}{2}\left(z_{c1} + z_{c2}\right) + \left(\mu \pm i\nu\right)\left(z_{c2} - z_{c1}\right) \tag{3.1}$$

where

$$\begin{aligned} \mu &= \frac{r_1^2 - r_2^2}{2d^2} \\ \nu &= \frac{1}{2d^2}\sqrt{4r_1^2 d^2 - \left(d^2 + r_1^2 - r_2^2\right)^2} \\ &= \frac{1}{2d^2}\sqrt{\left(r_1 + r_2 + d\right)\left(r_1 - r_2 - d\right)\left(-r_1 + r_2 - d\right)\left(r_1 + r_2 - d\right)} \end{aligned} \tag{3.2}$$

When the argument of the square root is zero the two circles intersect at a single point and when it is negative they do not intersect. The individual terms in the formula (3.1) have a simple interpretation. The first term on the right hand side is the midpoint between z_{c1} and z_{c2}. The second term represents corrections away from that point. The part proportional to μ gives a correction in a direction parallel to a line connecting z_{c1} to z_{c2} and the one proportional to ν is at right angles to it.

Application to Double Altitude Sights

The position of an observer can be determined from two altitude sights of the same celestial body taken at different times or of two different bodies taken at roughly the same time. In the former case, the effect of the observer's motion may need to be accounted for. This is done by applying a correction to one of the observed zenithal distances. The following example appears in ref.[7] (Ch.XVI, Ex.457, p.240). Two measurements are made of the Sun's altitude roughly 3 hours apart. At the first observation, the Greenwich Hour Angle (GHA), declination and zenithal distance of the Sun are

$$\mathrm{GHA}_1 = 6^{\mathrm{h}}45^{\mathrm{m}}58^{\mathrm{s}}.06$$

$$\delta_1 = -7°51'30''.3$$

$$\mathrm{ZD}_1 = 61°57'30''$$

and at the second observation

$$\mathrm{GHA}_2 = 9^{\mathrm{h}}49^{\mathrm{m}}11^{\mathrm{s}}.41$$

$$\delta_2 = -7°48'37''.3$$

$$\mathrm{ZD}_2 = 56°34'20''$$

Each observation defines a circle on the sphere giving a Line of Position (LOP). The observer's position lies at one of the intersections points of the LOP's. From eq.s (1.1) and (2.3) the first set of observations defines a circle with pole and angular radius

$$z_{p1} = \tan\left(\frac{\pi}{4} + \frac{\delta_1}{2}\right)e^{-i(\mathrm{GHA}_1)} = -0.173620 - 0.853988\,i, \qquad \rho_1 = \tan\left(\frac{ZD_1}{2}\right) = 0.600366\,.$$

Similarly the second set gives

$$z_{p2} = -0.733942 - 0.471227i, \qquad \rho_2 = 0.538132$$

Using eq.s(2.5) these correspond to circles on the complex plane with centers and radii

$$z_{c1} = -0.325224 - 1.599683i, \qquad r_1 = 1.454431$$

$$z_{c2} = -1.213898 - 0.779383i, \qquad r_2 = 1.215208$$

With $d = |z_{c1} - z_{c2}| = 1.209394$, eq.s(3.2) give

$$\mu = 0.218318, \qquad \nu = 0.964517$$

and eq.(3.1) shows that the circles intersect at

$$z_1 = -1.754769 - 1.867588i$$

$$z_2 = -0.172382 - 0.153304i$$

Only the former is a candidate for an observer in the northern hemisphere. From eq.(1.1) it is found that the observer's latitude and longitude are

$$L = 47°21'58''\mathrm{N}$$

$$\lambda = 133°12'58''\mathrm{W}$$

in agreement with ref. [7].

Altitude and Azimuth of a Celestial Body

The computation of altitude and azimuth represents a transformation from the equatorial coordinate system, given in terms of Greenwich Hour Angle (GHA) and declination, δ, to the observer's horizon coordinate system. As noted in the Appendix, rotations of spherical coordinate systems are realized by a class of simple bilinear transformations. In the horizon coordinate system the observer effectively occupies the pole and the required bilinear rotation therefore maps the observer's location to the zero on the complex plane. Taking into account that azimuth is measured by reference to north, the required rotation also places the North Pole on the positive real axis. For an observer at latitude, L, and longitude, λ, the coefficients for the required bilinear rotation (A.1) are

$$a = e^{-i\frac{\lambda}{2}}, \qquad b = \tan\left(\frac{\pi}{4} + \frac{L}{2}\right) e^{i\left(\frac{\lambda}{2} + \pi\right)} \tag{4.1}$$

that can be easily verified to satisfy the above requirements.

For a celestial body at Greenwich Hour Angle, GHA, and declination, δ, the geographic position (GP), i.e. the point on the Earth's surface at which the body is at the zenith, is at latitude $L = \delta$, longitude $\lambda = -\mathrm{GHA}$ corresponding to the point on the complex plane

$$z = \tan\left(\frac{\pi}{4} + \frac{\delta}{2}\right) e^{-i\,(\mathrm{GHA})}. \tag{4.2}$$

Performing the bilinear rotation results in a complex number, w,

$$\mathrm{T}(z) = \frac{az + b}{-\bar{b}z + \bar{a}} \equiv w = \tan\left(\frac{\pi}{4} - \frac{h}{2}\right) e^{iZ} \tag{4.3}$$

where h, Z are the body's altitude and azimuth at the observer's position. Note that a and b do not depend on the coordinates of the celestial body and hence, once determined, a and b can be used to compute LOP's from multiple objects.

The above results assume that a stereographic projection of the form (1.1) has been used. When the alternative stereographic projection (1.2) is used the above results become

$$z=\tan\left(\frac{\pi}{4}-\frac{\delta}{2}\right)e^{i\,(\mathrm{GHA})},\qquad a=e^{i\left(\frac{\lambda}{2}-\frac{\pi}{2}\right)},\qquad b=\tan\left(\frac{\pi}{4}-\frac{L}{2}\right)e^{i\left(-\frac{\lambda}{2}+\frac{\pi}{2}\right)}$$
$$\mathrm{T}(z)\equiv w=\tan\left(\frac{\pi}{4}-\frac{h}{2}\right)e^{iZ} \tag{4.4}$$

Example

The following example comes from the seminal paper by Saint-Hilaire [8] in which an altitude sight is taken of the star Vega (α Lyræ) at 8 o'clock on 24 October, 1874. In its original form longitudes and hour angles were measured from the meridian of Paris. Here the same numerical values will be used but with reference to Greenwich. The assumed position (AP) is at latitude and longitude

$$L=35°30'\mathrm{N},\qquad \lambda=9°30'\mathrm{W}\,.$$

The GHA and declination of the Sun are respectively

$$\mathrm{GHA}=62°16'00'',\qquad \delta=38°40'13''\,.$$

Converting these values to radian measure and substituting into eq.s (4.1) and (4.2) gives

$$a=0.99657+0.08281i,\qquad b=-1.93495+0.16078i,\qquad z=0.96846-1.84204i$$

Applying the bilinear rotation (A.1) yields

$$T(z)=0.13412-0.35573i=0.38017\,e^{-1.21026i}\,.$$

The altitude and azimuth can now be extracted from eq.(4.3) giving

$$h=48°22'08'',\qquad Z=290°39'.4\,(\equiv 69°20'.6\,\mathrm{W})\,.$$

Saint-Hilaire gives

$$h=48°22'15'',\qquad Z=69°20'\,\mathrm{W}$$

with the difference being due the number of significant figure carried at intermediate steps of his calculation.

When the alternative stereographic projection (1.2) is employed the numerical values are

$$a=-0.08281-0.99657i,\qquad b=-0.04265+0.51326i,\qquad z=0.22361+0.42531i$$

$$T(z)=0.13412-0.35573i=0.38017\,e^{-1.21026i}$$

Clearing Lunar Distance Sights

A lunar distance sight is concerned with the determining Greenwich Mean Time (GMT) that is subsequently used in finding longitude. The procedure is not dependent on the location of the observer and is largely an exercise in spherical trigonometry. As noted in the introduction, much of the simplicity and compactness of the foregoing results derives from the fact that both coordinates, L and λ, are subsumed into a single complex number. In the present case, without reference to a particular coordinate system, this advantage is not manifest. The methods can nevertheless be applied to derive formulas applicable to clearing lunars.

For the purposes of discussion, the other body involved in the lunar distance sight will be taken to be a star. Assume that the celestial sphere has been mapped to the complex plane by stereographic projection and subsequent bilinear rotations performed in such a way that the zenith is mapped to the origin, 0, and the observed position of the Moon is mapped to the point z_1 that lies on the positive real axis as shown in Fig.2.

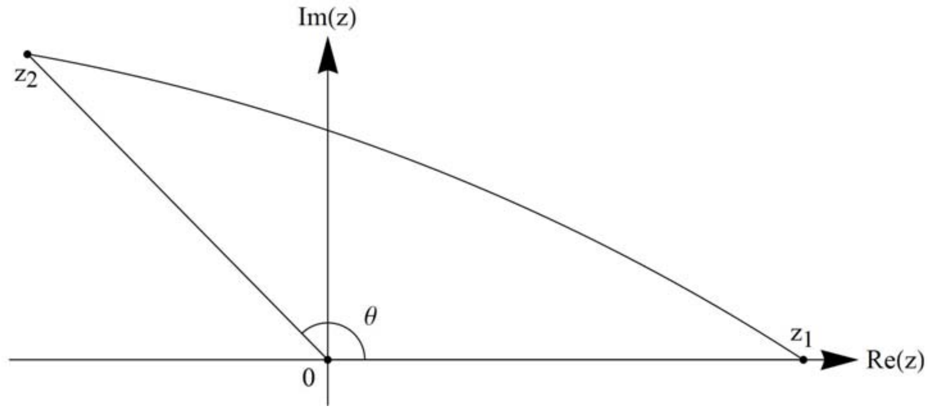

Figure 2. Image of a spherical triangle on the complex plane used in lunar distance sight clearing. The origin, 0, represents the zenith, z_1 the Moon and z_2 the star.

The observed position of the star is mapped to the point z_2. In this configuration, the straight line segments $\overline{0\,z_1}$ and $\overline{0\,z_2}$ are the images of segments of great circles on the sphere as is the arc $\overset{\frown}{z_1 z_2}$. Because stereographic projection is conformal, the angle $\angle z_1 0\, z_2$, denoted θ, is the true angular difference between the azimuthal angles of the Moon and the star. Let d, h_M, h_S denote the observed lunar distance and observed altitudes of the Moon and star respectively. From eq.(A.4) it follows that

$$\left| \frac{z_1 - |z_2| e^{i\theta}}{1 + z_1 |z_2| e^{-i\theta}} \right| = \tan\frac{d}{2} \tag{5.1}$$

in which

$$|z_1| = \tan\left(\frac{\pi}{4} - \frac{h_M}{2}\right); \qquad |z_2| = \tan\left(\frac{\pi}{4} - \frac{h_S}{2}\right) \tag{5.2}$$

To ease the notation the absolute value brackets will be dropped in what follows (i.e. $z_1 \equiv |z_1|$, $z_2 \equiv |z_2|$).

Using the identity $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$, eq.(5.1) yields

$$\tan^2\frac{d}{2} = \frac{z_1^2 + z_2^2 - 2z_1 z_2 \cos\theta}{1 + z_1^2 z_2^2 + 2z_1 z_2 \cos\theta} \tag{5.3}$$

and hence

$$\cos\theta = \frac{z_1^2 + z_2^2 - \left(1 + z_1^2 z_2^2\right)\tan^2\dfrac{d}{2}}{2z_1 z_2\left(1 + \tan^2\dfrac{d}{2}\right)} \tag{5.4}$$

Equations (5.3) and (5.4) can be used together for clearing lunar distance sights. This is best illustrated through an example. It is intriguing that in eq.(5.3) the sides of the triangle $z_1 0\, z_2$ all appear as the tangents of their half lengths.

Example

This example appears in ref.[9], (Ch.I, sec.20, p.27) that describes clearing a lunar distance using the star α Pegasi taken on 31st December, 1884 at Absarat, Nubia, Nile Valley.

Apparent lunar distance, $d = 103°26'24''$

Apparent lunar altitude, $h_M = 35°37'28''$

Apparent stellar altitude, $h_S = 40°17'24''$

Upon the application of corrections for horizon dip, refraction, parallax, augmentation etc., the geocentric altitudes are determined to be

Geocentric lunar altitude, $h'_M = 36°26'01''$

Geocentric stellar altitude, $h'_S = 40°16'15''$

Using in eq.s(5.2) with the apparent altitudes gives

$z_1 = 0.513661$

$z_2 = 0.463230$

and from the apparent lunar distance

$\tan^2 d/2 = 1.605615$

Plugging these results into eq.(5.4) yields

$\cos\theta = -0.982350$.

All the corrections necessary to obtain the geocentric altitudes of the Moon and star are corrections in altitude only and hence θ is unaffected by them. Applying eq.s(5.2) to the geocentric altitudes gives

$z'_1 = 0.504768$

$z'_2 = 0.463433$

which, when inserted into eq.(5.3) along with the previously obtained value for $\cos\theta$, produces the true geocentric distance of the Moon from α Pegasi at the time of measurement,

$\tan^2 d'/2 = 1.561277$

$\therefore d' = 102°39'30.6''$

The result given in ref.[9] is $d' = 102°39'30''$.

Conclusions

The foregoing paper has described the practical application of complex analysis to problems encountered in celestial navigation. Some classical examples of practical importance are treated to demonstrate the correctness of the results presented. The examples encompass the most common calculations performed in the course of celestial navigation but are not exhaustive. The methodology lends itself to the treatment of a wide range of problems involving spherical coordinate systems that arise in navigation and astronomy.

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Appendix

A complex number takes the form, $z = x + iy$, in which x is its real part and y is its imaginary part. The constant $i = \sqrt{-1}$. The same complex number can be expressed in a variety of equivalent forms

$$z = x + iy = r\left(\cos\phi + i\sin\phi\right) = re^{i\phi}$$

where r and ϕ are referred to as the modulus, and argument of z and written

$$|z| = r, \qquad \arg(z) = \phi.$$

The complex conjugate of z is denoted $\bar{z}$ and is given by

$$\bar{z} = x - iy = r\left(\cos\phi - i\sin\phi\right) = re^{-i\phi}$$

If the point P on the sphere maps, under stereographic projection, to z_p on the complex plane then the antipodal point P' maps to $z'_p = -1/\bar{z}_p$. This relationship forms the basis of the proof [10] that a rotation of the Riemann sphere is represented by a bilinear transformation (or bilinear rotation) of the form

$$\mathrm{T}(z) = w = \frac{az + b}{-\bar{b}z + \bar{a}} = -\frac{a}{\bar{b}} + \frac{1}{\bar{b}}\frac{\left(|a|^2 + |b|^2\right)}{\left(-\bar{b}z + \bar{a}\right)} \tag{A.1}$$

with inverse

$$\mathrm{T}^{-1}(w) = z = \frac{\bar{a}w - b}{\bar{b}w + a} \tag{A.2}$$

for complex constants a and b.

The composite of two successive rotations is easily obtained as

$$\mathrm{T}_2 \circ \mathrm{T}_1(z) = \frac{a_2\left(\dfrac{a_1 z + b_1}{-\bar{b}_1 z + \bar{a}_1}\right) + b_2}{-\bar{b}_2\left(\dfrac{a_1 z + b_1}{-\bar{b}_1 z + \bar{a}_1}\right) + \bar{a}_2} = \frac{\left(a_1 a_2 - \bar{b}_1 b_2\right)z + \left(b_1 a_2 + \bar{a}_1 b_2\right)}{-\left(\bar{b}_1\bar{a}_2 + a_1\bar{b}_2\right)z + \left(\bar{a}_1\bar{a}_2 - b_1\bar{b}_2\right)} \tag{A.3}$$

In ref.[3] it was shown that if P_1 and P_2 are points on the sphere with images z_{p1} and z_{p2} on the complex plane, the great circle distance, d, between the points is

$$d = 2\tan^{-1}\left|\frac{z_{p1} - z_{p2}}{1 + z_{p1}\bar{z}_{p2}}\right| \tag{A.4}$$

<u>Relationship of Bilinear Rotations to Other Formalisms</u>

The complex constants a and b in eq.(A.1) can be scaled by an arbitrary real constant without affecting $\mathrm{T}(z)$. They can therefore always be made to satisfy the unimodular condition, $|a|^2 + |b|^2 = 1$, and may then be identified with the Cayley-Klein parameters that arise in the 2×2 matrix representation of the Lie group SU(2) [11]. In terms of the 3-1-3 Euler angles $(\phi,\ \theta, \psi)$ commonly used to describe rotations in 3D space, a and b are

$$a = \cos\left(\frac{\theta}{2}\right)\cdot\exp\left(-i\,\frac{\phi+\psi}{2}\right); \quad b = -i\sin\left(\frac{\theta}{2}\right)\cdot\exp\left(i\,\frac{\phi-\psi}{2}\right) \tag{A.5}$$

Used in eq.(A.1), these expressions for a and b apply a coordinate system rotation equivalently expressed in terms of 3D rotation matrices as

$$\vec{w} \equiv \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{(A.6)}$$

or directly in the form of an SU(2) rotation as

$$\begin{pmatrix} z' & x'+iy' \\ x'-iy' & -z' \end{pmatrix} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \cdot \begin{pmatrix} z & x+iy \\ x-iy & -z \end{pmatrix} \cdot \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix} \qquad \text{(A.7)}$$

Rotations can also be performed using quaternions in which case the transformation equivalent to (A.6) and (A.7) is represented

$$\left(ix' + jy' + kz'\right) = \left(q_0 + iq_1 + jq_2 + kq_3\right) \cdot \left(ix + jy + kz\right) \cdot \left(q_0 - iq_1 - jq_2 - kq_3\right) \qquad \text{(A.8)}$$

where the q_i are real numbers satisfying the condition

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \qquad \text{(A.9)}$$

The quantities i, j, k are non-commuting under multiplication and satisfy the fundamental equations $i^2 = j^2 = k^2 = ijk = -1$. The quaternion components are related to a and b by

$$a = q_0 + iq_3; \quad b = -q_2 + iq_1 \qquad \text{(A.10)}$$

and thus may be extracted from their real and imaginary parts of upon enforcing the unimodular condition. Ref.[3] gives expressions for a and b in terms of parameters commonly used in astronomical applications. The quaternion representation of these transformations can thus be immediately derived using eq.s(A.10).

The components of the quaternion may also be represented as

$$\left(q_0 + iq_1 + jq_2 + kq_3\right) = \cos\frac{\Phi}{2} + \left(ie_1 + je_2 + ke_3\right)\sin\frac{\Phi}{2} \qquad \text{(A.11)}$$

and it can be shown that the quaternion product in eq.(A.8) executes a rotation by an angle Φ about the unit vector $\left(e_1, e_2, e_3\right)$.

As only 3 parameters are required to specify a rotation in 3D space, the four quaternion components form a redundant set. The redundancy is removed by defining the Rodriguez parameters, that form the components of the Gibbs vector, $\left(e_1, e_2, e_3\right)\tan\frac{\Phi}{2}$, at the cost introducing a singularity at $\Phi = 180^\circ$. The quaternion components, satisfying the condition (A.9), lie on a 4D hypersphere and it has been are shown [6] that the Rodriguez parameters are obtained by their stereographic projection in rectangular coordinates onto a 3D hyperplane tangent to the hypersphere. By adjusting the point of tangency, Modified Rodriguez parameters [5, 6] can be defined with a singularity $\Phi = 360^\circ$ or elsewhere. This procedure of adjusting the location of the singularity is analogous to choosing the point of tangency of the complex plane to the terrestrial globe to be at the South Pole, eq.(1.2), rather than the North Pole, eq.(1.1).

A survey of rotational mathematics can be found in ref.[12].